MACM 101 Final Exam

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There are a total of 100 points possible on this exam.

1. (6 points) Consider the statement "If \( x \) is a perfect square and \( x \) is even, then \( x \) is divisible by 4".

   (a) Designate propositional variables to stand for the three conditions about \( x \) mentioned in the statement.
   
   \[
   \begin{align*}
   S(x) & : \text{\( x \) is a perfect square} \\
   E(x) & : \text{\( x \) is even} \\
   D(x) & : \text{\( x \) is divisible by 4}
   \end{align*}
   \]

   (b) Write the statement formally in terms of these propositions.
   
   \[
   S(x) \land E(x) \implies D(x)
   \]

   (c) State the contrapositive of your answer in part (b), both in terms of your propositional variables and in colloquial terms.

   \[
   \neg D(x) \implies \neg S(x) \lor \neg E(x)
   \]

   Easy.
2. (10 points) Six different numbers were chosen at random from the numbers 1 through 49. The winning combinations do not depend on the order in which these numbers are drawn.

(a) How many different lottery outcomes are possible? \[ \binom{49}{6} \]

(b) A jackpot prize occurs if all numbers are chosen correctly. What is the probability of winning the jackpot? \[ \frac{1}{\binom{49}{6}} \]

(c) If you choose five out of six correctly, you share the second prize. What is the probability of winning the second prize? \[ \frac{\binom{6}{5} \times 43}{\binom{49}{6}} \]
3. (10 points)

(a) In how many ways can the letters in UNUSUAL be arranged?

\[
\frac{7!}{3!} \]

(b) For the arrangements in part (a), how many have all three U's together?

\[
5! \]

(c) How many of the arrangements in part (a) have no two consecutive U's.

\[
\frac{4!}{3!} (5) \]
4. (6 points) For any positive integer \( n \) show by using the binomial theorem that

\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k 2^{n-k} = 1.
\]

\[
\text{Binomial Expansion:} \quad (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \cdot (-1)^k
\]

Replace \( x = 2 \) and \( y = 1 \).

5. (4 points) There must be something wrong with the following induction proof; What is it?

Theorem: For all positive integers \( n \), \( 2^{n-1} = 1 \).

Proof. If \( n = 1 \), \( 2^{1-1} = 2^0 = 1 \). Suppose that the theorem is true for all \( n \leq k \). Now we have

\[
2^{(k+1)-1} = 2^k = \frac{2^{k-1} \cdot 2^{k-1}}{2^{k-2}} = \frac{1 \times 1}{1} = 1.
\]

Therefore, the theorem is true for \( n = k + 1 \) as well. Hence the theorem is true for all positive integers (Using the principle of strong mathematical induction).

Not true when \( k + 1 = 2 \).
6. (10 points) Prove by induction the following generalization of De Morgan's law to $n$ sets.

$$A_1 \cup A_2 \cup \ldots \cup A_n = A_1 \cap A_2 \cap \ldots \cap A_n$$

**Already discussed.**

7. (5 points) How many ordered pairs of integers are needed to guarantee that there are two ordered pairs $(a_1, b_1), (a_2, b_2)$ such that $a_1 + a_2$ is even and $b_1 + b_2$ is even?

**Answer:**

There are four ordered pairs $(a, b)$ whose parities are

- $(odd, odd)$,
- $(odd, even)$,
- $(even, odd)$,
- $(even, even)$

If 5 ordered pairs are selected, there exist two ordered pairs with the same parities.
8. (8 points) There are 51 houses on a street. Each house has an address between 1 and 100 inclusive.

(a) Show that at least two houses have addresses that are consecutive.

\[
\begin{align*}
\text{Pigeons} & = 51 \\
\text{# of holes} & = 50 \\
\text{Define } f : \text{pigeons} \rightarrow \text{holes where } \\
f(n) & = \left\lfloor \frac{n}{2} \right\rfloor
\end{align*}
\]

(b) Show that at least two houses have addresses such that the sum of their addresses are divisible by 100.

You need to pick 52 houses. The pigeonholes are:

\[
\{1, 99\}, \{2, 98\}, \ldots, \{49, 51\}, \{50\}, \{100\}.
\]
9. (10 points)

(a) Let $R$ be a relation defined on $A \times B$ such that $((a, b), (z, y)) \in R$ if and only if $a \leq z$ and $b \leq y$. Show that $R$ is a partial order relation.

Done in the class/tutorial.

(b) Draw the Hasse diagram for the poset $(A \times B, R)$ where $A = \{1, 2, 3\}$ and $B = \{2, 3\}$ and $R$ is defined as in part (a).
10. (5 points) Let the relation $R$ be reflexive and transitive on $A$. Show that $R \cap R^{-1}$ is an equivalence relation on $A$.

11. (8 points) Determine whether each of the following statements is true or false. For each false statement give a counterexample.

(a) If $f : A \to B$ and $(a, b), (a, c) \in f$, then $b = c$.

True

(b) If $f : A \to B$ is a one-to-one correspondence and $A$ and $B$ are finite, then $A = B$.

Same as bijective, size should be equal

(c) If $f : A \to B$ is one-to-one, then $f$ is invertible.

Not true

$f$ is not invertible
(d) $f : A \rightarrow B$ and $A_1, A_2 \subseteq A$, then $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.

True

(e) $f : A \rightarrow B$ and $B_1, B_2 \subseteq B$, then $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

True

12. (8 points) Provide a recursive definition for each of the following languages $A \subseteq \Sigma^*$ where $\Sigma = \{0, 1\}$.

(a) $x \in A$ if (and only if) the number of 0's in $x$ is even.

(b) $x \in A$ if (and only if) $x = x^R$ where $x^R$ is the reversal of $x$. (The reversal of 101100 is 001101.)

Ignore
13. (10 points)

Let $I = \{0, 1, 2\}$ and $O = \{0, 1\}$ be the input and the output alphabet respectively. A string $x \in I^*$ is said to have the odd parity if it contains an odd number of 1's and odd number of 2's. Construct a finite state machine that recognizes all nonempty string of odd parity.