MACM 101 : Homework 4 (October 21, 2019)
Homework is due by Oct.30, 2019
Exercises on Sets and Probability.

The lecture slides file contains a few problems from the text. Here are some additional problems.

1. Write the following set by listing their elements between braces.
   (a) $\{x \in \mathbb{R} : x^2 = 7\}$
   $\{-\sqrt{7}, \sqrt{7}\}$
   (b) $\{x \in \mathbb{Z} : | 2x | < 5\}$
   $\{-2, -1, 0, 1, 2\}$
   (c) $\{x \in \mathbb{Z} : -2 < x \leq 7\}$
   $\{-1, 0, 1, 2, 3, 4, 5, 6, 7\}$
   (d) $\{x \in \mathbb{Z} : -5 < x \leq 2\}$
2. Write the following set in set-builder notation.
   (a) $\{-3, -2, -1, 0, 1, 2, 3\}$
   $\{x : -3 \leq x \leq 3\}$
   (b) $\{0, 1, 8, 27, 64, 125, \dots\}$
   $\{x^3 : x \in \mathbb{N}\}$
   (c) $\{0, -1, -4, -9, \dots\}$
   $\{-x^2 : x \in \mathbb{N}\}$
3. Let $\mathbb{R}$ be the universal set. Let $A = \{1\}$, $B = (0,1) = \{x : 0 < x < 1\}$ and $C = [0,1] = \{x : 0 \leq x \leq 1]$. Write down the following sets.
   - $A \cup B$; $A \cap B$; $B \cap C$; $A \cup C$; $A \cap C$
   (0,1] ϕ (0,1) [0,1] $\{1\}$

   Are any of the pairs of sets A, B and C disjoint? A and B are disjoint.
4. Let A, B, C, D be nonempty sets. Prove that $A \times B \subseteq C \times D$ if and only if $A \subseteq C$ and $B \subseteq D$.

   We can prove this by showing $A \times B \subseteq C \times D \Rightarrow (A \subseteq C) \wedge (B \subseteq D)$ and $(A \subseteq C) \wedge (B \subseteq D) \Rightarrow A \times B \subseteq C \times D$.

   Case (a): $A \times B \subseteq C \times D \Rightarrow (A \subseteq C) \wedge (B \subseteq D)$
   Consider an arbitrary element $(x,y) \in A \times B$, where $x \in A$ and $y \in B$. Since $A \times B \subseteq C \times D$, therefore, $(x,y) \in C \times D$, i.e. $x \in C$ and $y \in D$. Thus we have established that an arbitrary element x of A is also an element of C. Therefore, $A \subseteq C$. Similarly, we can conclude that $B \subseteq D$.

   Case (b): $(A \subseteq C) \wedge (B \subseteq D) \Rightarrow A \times B \subseteq C \times D$ Follow the arguments in (a) in reverse order.

5. Let A, B and C be three arbitrary subsets of the universal set U. Use an element containment proof (i.e. prove that the left side is a subset of the right side and the right side is a subset of the left side) to prove the following:

   - $\overline{A \cap B \cup C} = \overline{A} \cup \overline{B} \cap \overline{C}$.
     We will use the precedence rule to evaluate the expression. Thus we need to show that $\overline{(A \cap B) \cup C} = (\overline{A} \cup \overline{B}) \cap \overline{C}$. This can be easily established using the DeMorgan's Law.
   - $\overline{A \cup B \cap C} = \overline{A} \cap \overline{B} \cup \overline{C}$.
     The proof is similar.

6. Use the membership table method to determine which membership $\subseteq, =, \supseteq$ is true for the following pair of sets.

   - $(B - C)$, $\quad (B - A) - (C - A)$.

A	B	C	(B-A)	(B-C)	(C-A)	(B-A)-(C-A)
0	0	0	0	0	0	0
0	0	1	0	0	1	0
0	1	0	1	1	0	1
0	1	1	1	0	1	0
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	0	1	0	0
1	1	1	0	0	0	0

7. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

   Let $(x, y) \in A \times (B \cap C)$
   $\Rightarrow x \in A \wedge y \in B \cap C$
   $\Rightarrow x \in A \wedge (y \in B \wedge y \in C)$
   $\Rightarrow (x \in A \wedge y \in B) \wedge (x \in A, y \in C)$
   $\Rightarrow (x, y) \in A \times B \wedge (x, y) \in A \times C$
   $\Rightarrow (x, y) \in (A \times B) \cap (A \times C)$
   So $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.

   Similarly let $(x, y) \in (A \times B) \cap (A \times C)$
   $\Rightarrow (x, y) \in A \times B \wedge (x, y) \in A \times C$
   $\Rightarrow x \in A, y \in B \wedge x \in A, y \in C$
   $\Rightarrow x \in A, y \in B \wedge y \in C$
   $\Rightarrow x \in A, y \in B \cap C$
   $\Rightarrow (x, y) \in A \times (B \cap C)$
   So $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.
   Since both sets are subsets of one another , so $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

8. Two fair six-sided dice are rolled and the sum s of the numbers coming up is recorded. What is the probability of $s \geq 10$? Show your work.

Let S be the sample space of the experiment of throwing two distinguished dice (red and blue). Then $S = \{(1,1),(1,2)...,(1,6),(2,1),(2,2),...,(2,6),........,$ $(6,11),(6,2),...,(6,6)\}$. The cardinality of S is 36. The probability of an outcome is $\frac{1}{6}$. Let $A \subseteq S$ where $A = \{(a,b) \in S | a+b \geq 10\}$. Hence $A = \{(4,6),(5,5),$ $(6,4),(5,6),(6,5),(6,6)\}$. Therefore, $Pr(A) = \frac{|A|}{|S|} = \frac{6}{36}$.

When the pair of dice is indistinguishable, The sample space is $S = \{< 1,1 >, < 1,2 >, \ldots < 1,6 >, < 2,2 >, < 2,3 > \ldots, < 2,6 >, ..., < 6,6 >\}$. The event set is $A = \{< 4,6 >, < 5,5 >, < 5,6 >, < 6,6 >\}$. In this case, $Pr(A) = \frac{4}{21}$.

9. A random experiment consists of rolling an unfair, six-sided die. The digit 6 is three times as likely to appear as the numbers 2 and 4. The numbers 2 and 4 are twice as likely to appear as one of the numbers, 1, 3, and 5.

   Assign appropriate probabilities to the six outcomes in the sample space.

   Since the die is unfair, the probability of the outcomes are not the same. Let p be the probability of each of the outcomes of 1, 3 and 5. Then the probability of each of the outcomes of 2 and 4 is $2p$. The the probability of outcome 6 is $6p$. Since $6p+4p+p+p+p = 1$, therefore $p = \frac{1}{13}$. Therefore, $Pr(\{1\})Pr(\{3\}) = Pr(\{5\}) = 1/13$. $Pr(\{2\}) = Pr(\{4\}) = \frac{2}{13}$. Lastly, $Pr(\{6\}) = \frac{6}{13}$.