MACM 101 Quiz 1 (in the class)
October 16, 2019.

Part A: Answer questions worth 20 points

1. (10 points) Consider a pack of 50 cards which consists of cards numbered $1, 2, \dots, 10$ in red, yellow, green, magenta and blue. We are interested in an arrangement of these cards.

   Find the number of ways to obtain a 5-card arrangement where

   Some students might treat this problem as a combination problem. This will be incorrect. The term "arrangement" is associated with ordering. If the solution approach to the problem is addressed as a combination problem, please correct them as a combination problem. Deduct 30% for the incorrect assumption. There are four parts. Each part is worth 2.5.

   (a) all arrangements are valid,
   Solution: $P(50,5)$ (order is important)

   (b) only arrangements with all the same colour cards,
   Solution: For red coloured cards, there are $P(10,5)$ 5-card sequences. The answer is $5 \times P(10,5)$ for all 5 colors.

   (c) only arrangements with two or more cards having the same number, and
   Solution: Let us count all 5-card sequences where all the cards have distinct numbers. The number of such sequences is $50 \times 45 \times 40 \times 35 \times 30$. After selecting the first card, we are not allowed to pick any card with the same number as the first card. We use the same strategy to select the remaining cards. The answer to the question is, therefore, $P(50,5) - 50 \times 45 \times 40 \times 35 \times 30$.

   (d) only arrangements that start with 4 and end with 5.
   Solution: There are $P(5,1)$ (or $C(5,1)$) ways to select 4 and $P(5,1)$ ways to select 5. The other three cards could be anything. The answer is, therefore, $P(5,1)^2 . P(48,3)$.

2. (10 points) Consider the problem of determining the number of integral solutions to the following equation:

   $$x_1 + x_2 + x_3 + x_4 + x_5 = 150,$$

   There are two parts. Each part is worth 5 points.

(a) where $1 \leq x_i, \ \forall\, i$,
Solution: The answer is $C(150-5+(5-1),(5-1))$.

(b) where for each $i, \ 1 \leq i \leq 5$, x_i is a multiple of 6.

In the absence of no condition on x_i (≥ 0 or ≥ 1), any assumption a student makes should be treated as correct.

Solution: Suppose we write $x_i = 6.y_i, i = 1,2,3,4,5$. The problem can then be restated as:

Determine the number of integral solutions to the following equation:

$$6y_1+6y_2+6y_3+6y_4+6y_5 = 150, \quad 1 \leq y_i(\textit{when } x_i \geq 1 \textit{ is the assumption}), \ \ \forall\, i \,.$$

which is the same as

$$y_1+y_2+y_3+y_4+y_5 = 25, \quad 1 \leq y_i, \ \ \forall\, i \,.$$

The answer is $C(20+5-1,5-1) = C(24,4)$.

When $x_i \geq 0$ is the assumption, the answer is $C(25+5-1,5-1) = C(29,4)$

3. (10 points) Consider the expansions of $(1+x)^8$ and $(1-2x)^7$.

   (a) Determine the coefficient of x^5 in both the expansions.

   (b) What is the coefficient of x^5 in $(1+x)^8-(1-2x)^7$?

   (c) What is the sum of all the coefficients of the binomial expansion of $(1+x)^8-(1-2x)^7$?

10 marks are distributed as: 2.5 marks each for (a) and (b), and 5 marks for (c). Any solution to (c) that writes the coefficients of $(1+x)^8-(1-2x)^7$ explicitly should be penalized by at least 2.5 marks. Full marks are only given when it is recognized that by plugging $x = 1$ one can realize the solution. **Solution:** The coefficient of x^5 in the binomial expansion of $(1+x)^8$ is $\binom{8}{5}.(1)^5$. The coefficient of x^5 in the binomial expansion of $(1-2x)^7$ is $\binom{7}{5}.(-2)^5$. The answer is, therefore $\binom{8}{5}-\binom{7}{5}2^5$.

For the third subproblem, we set $x = 1$. The sum is then $2^8-(-1)^7$.

Part B: Answer questions worth 20 points

1. (10 points) Write the following arguments in symbolic form. Then establish the validity of the argument.

   If Sandy gets the manager's position and works hard, then she will get a raise. If she gets the raise, she will buy a new truck. She has not purchased a new truck. Therefore either Sandy did not get the manager's position or she did not work hard.

   4 marks should be reserved for writing the arguments in symbolic form. The second part has 6 marks. The argument must mention the inference rule being applied explicitly. Otherwise, remove 1/2 mark for each violation. **Solution:** Symbolic equivalence:

   p: Sandy gets manager's position;
   q: Sandy works hard;
   r: sandy gets a raise;
   s: Sandy buys a new car;

   The arguments are:

   $(p \wedge q) \rightarrow r$
   $r \rightarrow s$
   $\neg s$
   ____
   $\neg p \vee \neg q$

   Validity of the argument

(1)	$\neg s$	Premise
(2)	$r \rightarrow s$	Premise
(3)	$\neg r$	Steps (1), (2) and Modus Tollens
(4)	$(p \wedge q) \rightarrow r$	Premise
(5)	$\neg(p \wedge q)$	Steps (3), (4) and Modus Tollens
(6)	$\therefore \neg p \vee \neg q$	Step (5) and DeMorgan's law

2. (10 points) Show that the following statements are equivalent. The universe is the set of all integers. The answer should first formulate the problem as a logic problem.

   (a) $p_1(n)$: n^2 is an odd integer.

   (b) $p_2(n)$: n is an odd integer.

(c) $p_3(n)$: $(n+1)$ is an even integer.

Soloution: The answer is complete if we can show that $p_2(n) \to p_3(n)$, $p_3(n) \to p_1(n)$, and $p_1(n) \to p_2(n)$.

- $p_2(n) \to p_3(n)$: Direct proof. n is odd implies $n+1$ is even.
- $p_3(n) \to p_1(n)$: Direct proof. $n+1$ even implies n is odd. Thus, n^2 is odd.
- $p_1(n) \to p_2(n)$: Contrapositive proof: Show $\neg p_2(n) \to \neg p_1(n)$, i.e "if n is even, n^2 is even". This is obviously true.

The above proof is equivalent to showing $p_1(n) \leftrightarrow p_2(n)$, $p_2(n) \leftrightarrow p_3(n)$ and $p_3(n) \leftrightarrow p_1(n)$.

Deduct some marks if the proof is not if and only if. Also deduct some marks if the problem is not formulated as a logic problem.

3. (10 points) Let n be an odd integer. Prove that n^3+2n^2 is also odd using the following proof methods. The answer must start with the formulation of the problem, given the proof method.
solution Let $p : n$ is an odd integer; $q : n^3+2n^2$ is odd. We need to show $p \to q$. Each part is worth 3 points. They get 10 if all three are answered correctly.

   (a) Direct proof.
   Let odd $n = 2.t+1, t$ an integer. Then $(2t+1)^3+2(2t+1)^2 = (8t^3+12t^2+6t+1)+2(4t^2+4t+1) = 2(4t^3+10t^2+7t+1)+1$. This implies that q is true.

   (b) Contrapositive (indirect) proof.
   We need to show that $\neg q \to \neg p$, i.e., we need to show that n^3+2n^2 is even implies n is even. Since $2n^2$ is even, n^3+2n^2 is even implies n is even.

   (c) Proof by contradiction.
   Here we assume that p is true and $\neg q$ is true. n^3+2n^2 is even implies that n^3 is even. Therefore, n is even, i.e. $\neg p$ is true. Thus we have p is true and $\neg p$ is true. We arrive at a contradiction. Therefore, our assumption the $\neg q$ is true is false. Therefore $p \to q$ is true.