MACM 101 : Quiz 4 (Tuesday- D1) (Full Marks 22) Time: 40 minutes

1. (9 points) Consider the following theorem: If a is an odd number, a^2+2a+5 is even.

   (a) Give a direct proof of the theorem.

   Suppose a is an odd number, a^2 is odd, $2a$ is even, 5 is odd, thus a^2+2a+5 is even.

   (b) Give a contrapositive proof of the theorem.

   Suppose a^2+2a+5 is odd, we know $2a$ is even, 5 is odd, so a^2 is even. Then a is even

   (c) Give a proof by contradiction of the theorem.

   a is an odd number, suppose a^2+2a+5 is odd, then we have a^2 is odd, $2a$ is even. Then 5 must be an even number in order to make a^2+2a+5 odd. We have a contradiction since 5 is odd number.

2. (5 points) Prove the following theorem.

   - For all positive integers n: n is even if and only if $3n^2+8$ is even.

     Suppose n is even, then n^2 is even, $3n^2$ is even, 8 is even, so $3n^2+8$ is even
     Suppose $3n^2+8$ is even, we know 8 is even, so $3n^2$ is even. Thus n is even.

   - Given an integer n, then n^3+n^2+n is even if and only if n is even.

     Suppose n is even, then n^3, n^2,n all even. Thus n^3+n^2+n is even.
     Suppose n^3+n^2+n is even, since n^3, n^2,n are of the same parity. Then n is even.

   - Prove that $\sqrt{2}$ is irrational.

     Proof by contradiction Suppose $\sqrt{2}$ is rational. Then $\sqrt{2}=a/b$ for some integer a,b
     By squire both sides of the equation we have $a^2=2b^2$
     Then a is even. Suppose $a=2k$ for some integer k

Then we have $4k^2 = 2b^2$, Thus $b = 2k^2$, Then b is even.
We derive a contradiction that a and b have common factor 2.
Thus $\sqrt{2}$ is irrational.

3. (5 points) Show that the following statements are equivalent.

   - p_1 : n is an even integer
   - p_2 : $(n+1)$ is an odd integer
   - p_3 : n^2 is an even integer.

   If n is an even integer, then $n+1$ is an odd integer. If $n+1$ is an odd integer, n is even. So p_1 and p_2 are equivalent.
   If n is an even integer, then n^2 is an even integer. If n^2 is an even integer, n is even. So p_1 and p_3 are equivalent.
   So p_1, p_2,p_3 are equivalent.

4. (3 points) Disprove by counterexample the following proposition.

   - The product of two irrational numbers is always irrational.

     $\sqrt{2}$ is irrational number, and $\sqrt{2}^2 = 2$ which is rational.

   - If $x, y \in \mathbb{R}$, then $|x+y| = |x| + |y|$.

     Let $x = 1$ and $y = -1$. Then $|x+y| = 0$ but $|x| + |y| = 2$

5. (Bonus Question) (5 points)

   - If p is a prime and $0 < k < p$, $p | \binom{p}{k}$.

     We know $\frac{p!}{k!(p-k)!} = \binom{p}{k} = \frac{1*2*...*p}{1*2*...*k*1*2*...*(p-k)}$, since $k < p$, all the factors in the denominator less than p, so they do not cancel the p in the numerator. So $p | \binom{p}{k}$.

   - If $n \in \mathbb{N}$, $\binom{2n}{n}$ is even.

     We are asked to choose N items from a set of $2N$ items. Suppose the items we choose form a set A. Then $2N - A$ is also a solution. Thus the solution comes in pairs. $\binom{2n}{n}$ is even.