1. (9 points) Consider the following theorem: If \(a\) is an odd number, \(a^2 + 2a + 5\) is even.

(a) Give a direct proof of the theorem.

Suppose \(a\) is an odd number, \(a^2\) is odd, \(2a\) is even, 5 is odd, thus \(a^2 + 2a + 5\) is even.

(b) Give a contrapositive proof of the theorem.

Suppose \(a^2 + 2a + 5\) is odd, we know \(2a\) is even, 5 is odd, so \(a^2\) is even. Then \(a\) is even.

(c) Give a proof by contradiction of the theorem.

\(a\) is an odd number, suppose \(a^2 + 2a + 5\) is odd, then we have \(a^2\) is odd, \(2a\) is even. Then 5 must be an even number in order to make \(a^2 + 2a + 5\) odd. We have a contradiction since 5 is odd number.

2. (5 points) Prove the following theorem.

- For all positive integers \(n\): \(n\) is even if and only if \(3n^2 + 8\) is even.

Suppose \(n\) is even, then \(n^2\) is even, \(3n^2\) is even, 8 is even, so \(3n^2 + 8\) is even

Suppose \(3n^2 + 8\) is even, we know 8 is even, so \(3n^2\) is even. Thus \(n\) is even.

- Given an integer \(n\), then \(n^3 + n^2 + n\) is even if and only if \(n\) is even.

Suppose \(n\) is even, then \(n^3, n^2, n\) all even. Thus \(n^3 + n^2 + n\) is even.

Suppose \(n^3 + n^2 + n\) is even, since \(n^3, n^2, n\) are of the same parity. Then \(n\) is even.

- Prove that \(\sqrt{2}\) is irrational.

Proof by contradiction Suppose \(\sqrt{2}\) is rational. Then \(\sqrt{2} = a/b\) for some integer \(a,b\)

By squire both sides of the equation we have \(a^2 = 2b^2\)

Then \(a\) is even. Suppose \(a = 2k\) for some integer \(k\)
Then we have $4k^2 = 2b^2$. Thus $b = 2k^2$, Then $b$ is even. We derive a contradiction that $a$ and $b$ have common factor 2. Thus $\sqrt{2}$ is irrational.

3. (5 points) Show that the following statements are equivalent.
   - $p_1 : n$ is an even integer
   - $p_2 : (n + 1)$ is an odd integer
   - $p_3 : n^2$ is an even integer.

   If $n$ is an even integer, then $n + 1$ is an odd integer. If $n + 1$ is an odd integer, $n$ is even. So $p_1$ and $p_2$ are equivalent.
   If $n$ is an even integer, then $n^2$ is an even integer. If $n^2$ is an even integer, $n$ is even. So $p_1$ and $p_3$ are equivalent.
   So $p_1, p_2, p_3$ are equivalent.

4. (3 points) Disprove by counterexample the following proposition.
   - The product of two irrational numbers is always irrational.

   \[ \sqrt{2} \text{ is irrational number, and } \sqrt{2}^2 = 2 \text{ which is rational.} \]
   - If $x, y \in \mathbb{R}$, then $|x + y| = |x| + |y|$.

   Let $x = 1$ and $y = -1$. Then $|x + y| = 0$ but $|x| + |y| = 2$

5. (Bonus Question) (5 points)
   - If $p$ is a prime and $0 < k < p$, $p|\binom{p}{k}$.

   We know $\frac{p!}{k!(p-k)!} = \binom{p}{k} = \frac{1 \cdot 2 \cdot \ldots \cdot p}{1 \cdot 2 \cdot \ldots \cdot k \cdot 1 \cdot 2 \cdot \ldots \cdot s \cdot (p-k)}$, since $k < p$, all the factors in the denominator less than $p$, so they do not cancel the $p$ in the numerator. So $p|\binom{p}{k}$.
   - If $n \in \mathbb{N}$, $\binom{2n}{n}$ is even.

   We are asked to choose $N$ items from a set of $2N$ items. Suppose the items we choose form a set $A$. Then $2N - A$ is also a solution. Thus the solution comes in pairs. $\binom{2n}{n}$ is even.