October 9-10 Tutorial - Solutions

Quantified propositions $\rightarrow$ English statements.

1. **Question 6, section 1.4 of Rosen**
   Let $N(x)$ be the statement "x has visited North Dakota" where the domain consists of the students in your school. Express each of these quantifications in English.

   a) $\exists x\ N(x)$
   Solution: "There is a student in my school who has been to North Dakota."

   b) $\forall x\ N(x)$
   Solution: "All students in my school have been to North Dakota."

   c) $\neg\exists x\ N(x)$
   Solution: "No one in my school has been to North Dakota."

   d) $\exists x\ \neg N(x)$
   Solution: "There is a student in my school who has never been to North Dakota."

English statements $\rightarrow$ quantified propositions.

1. Question 10, section 1.4 of Rosen.
   Let $C(x)$ be the statement "x has a cat," let $D(x)$ be the statement "x has a dog," and let $F(x)$ be the statement "x has a ferret." Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

   (a) A student in your class has a cat, a dog, and a ferret.
   Solution: $\exists x\ (C(x) \wedge D(x) \wedge F(x))$

   (b) All students in your class have a cat, a dog, and a ferret.
   Solution: $\forall x\ (C(x) \wedge D(x) \wedge F(x))$

   (c) Some student in your class has a cat and a ferret, but not a dog.
   Solution: $\exists x\ (C(x) \wedge F(x) \wedge \neg D(x))$

2. Question 42, section 1.4 of Rosen.
   Express each of these system specifications using predicates, quantifiers, and logical connectives.

   a) Every user has access to an electronic mailbox.
   Solution: We can limit the domain to all users of the system, and introduce the E predicate:

   $$E(x, e) := \text{user } x \text{ has access to electronic mailbox } e$$

   Therefore, we're quantifying over two domains in the E predicate, the domain of system users and the domain of electronic mailboxes. This requires that we quantify over both domains:

   $$\forall x\ \exists e\ M(x, e)$$

   b) The system mailbox can be accessed by everyone in the group if the file system is locked.
   Solution: You may be tempted to create a predicate quantifying over all file systems here, but we're referring to a single file system, in the definite sense. It's the file system state that is uncertain, and which therefore needs to be quantified over. The states it can have are: locked, unlocked, ...(possible further states of which we have no knowledge). So, we introduce the predicate

   $$F(s) := \text{ the file system is in state } s$$

   For users in the group, we have the predicate

   $$G(x) := \text{ user } x \text{ is in the group}$$

   and for system mailbox access, we have

   $$SM(x) := \text{ user } x \text{ can access the system mailbox}$$

   Now to write the actual statement. We want to ensure that the user x is in the group, and we want the condition that the file system is locked to hold before we assert that users in the group can access the system mailbox. This leads us to

   $$\forall x\ G(x) \wedge F(\text{locked}) \rightarrow SM(x)$$

   Notice that "locked" is a constant, representing the "locked" state of the filesystem (indeed, there is no propositional variable named "locked" we've quantified over! Any variable we introduce must be quantified).

c) The firewall is in a diagnostic state only if the proxy server is in a diagnostic state.
Solution: The tricky part here is that the firewall has any number of states, of which we know nothing, and several of them could be considered "diagnostic" states. Likewise with the proxy server. Again, we're referring to the firewall and the proxy each in the definite sense, so they should not be quantified over with variables. Instead, let's quantify over the domain of firewall and proxy states, and let's introduce a predicate to identify diagnostic states:

$$D(x) := \text{state } x \text{ is a diagnostic state}$$

and let's introduce the predicates P and F:

$$F(x) := \text{the firewall is in state } x$$

$$P(x) := \text{the proxy is in state } x$$

From this, I was immediately led to write

$$\forall x\ D(x) \land (F(x) \rightarrow P(x))$$

but this is wrong. Why? Because the original statement allows the firewall and the proxy server to be in distinct diagnostic states, and the implication should still hold. The use of the same variable means that the proxy server is in a diagnostic state if firewall is in that exact same diagnostic state. In other words, I succumbed to the mistake described in the final part of question 2, above. We need distinct variables for these states, and they must both be diagnostic states:

$$\forall x\ \forall y\ D(x) \land D(y) \land (F(x) \rightarrow P(y))$$

d) At least one router is functioning normally if the throughput is between 100 kbps and 500 kbps and the proxy service is not in diagnostic mode.
Solution: Again, the proxy service can have a number of modes, and we create a predicate to assert that the proxy service is in mode m:

$$P(m) := \text{the proxy server is in mode } m$$

The remaining predicates we need are straightforward:

$$R(r) := \text{router } r \text{ is functioning normally}$$

$$T(m, n) := \text{the throughput is between } m \text{ kbps and } n \text{ kbps}$$

$$D(m) := m \text{ is the diagnostic mode}$$

Since we are quantifying on at least one router that is functioning normally, the use of $\exists$ suffices over the domain of routers. Similarly to the previous question, then, the final proposition is

$$\forall m\ \exists r\ D(m) \land [(T(100, 500) \land \neg P(m)) \rightarrow R(r)]$$

Contradicting quantified propositions.

1. Question 36, section 1.4 of Rosen.
Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all real numbers.

a) $\forall x\ (x^2 \neq x)$
Solution: $x = 1$ and $x = 0$ are all the counterexamples, which is to say that $x^2 = x$ in either case.

b) $\forall x\ (x^2 \neq 2)$
Solution: $x = \sqrt{2}$ and $x = -\sqrt{2}$ are the only two counterexamples.

c) $\forall x\ (|x| > 0)$
Solution: $x = 0$ is the one counterexample.