MACM 101 : Tutorial (November 13-14, 2019)

1. Answer the following short questions.

   (a) Give an example of a function f from integers to integers ($f : Z \to Z$) that is onto but not one- to-one.

   Solution: The function $f : Z \to Z$ defined by $f(n) = \lfloor \frac{n}{2} \rfloor$ is onto but not one-to-one.

   (b) Does the formula $f(x) = \frac{1}{x^2-2}$ define a function $f : R \to R$? A function $f : Z \to R$?

   Solution: When the domain of f is R, f is not a function since $f(\sqrt{2})$ is not defined. When the domain of f is N, f is a function.

   (c) **Inclass**
   Prove by cases that for any positive integer a, $\lfloor \frac{a}{2} \rfloor + \lceil \frac{a}{2} \rceil = a$.

   Solution: The proof is "by cases". When a is an even integer, i.e. $a = 2k$, then $\lfloor \frac{a}{2} \rfloor + \lceil \frac{a}{2} \rceil = k + k = 2k = a$.
   When $a = 2k+1$ (i.e. odd), then $\lfloor \frac{a}{2} \rfloor + \lceil \frac{a}{2} \rceil = k+k+1 = 2k+1 = a$.

   (d) If $A = \{1,2,3,4,5\}$ and $B = \{x,y,z\}$, determine the number of one-to-one, onto and constant functions one can design.

   Solution: Number of one to one function = 0 (since $|A| > |B|$). Let A represent a set of guests and B represents a set of rooms.

   - There are 3^5 ways to distribute the guests to rooms.
   - Out of these, $C(3,1) \times 2^5$ ways to distribute the guests with at least one room vacant.
   - $C(3,2) \times 1^5$ ways to distribute the guests with two vacant rooms.
   - There are $C(3,1) \times 2^5 - C(3,2) \times 1^5$ ways to have a vacancy.
   - Hence there are $S(5,3) = 3^8 - (C(3,1) \times 2^5 - C(3,2) \times 1^5) = 3^8 - C(3,1) \times 2^5 + C(3,2) \times 1^5$ ways to occupy all rooms.

   (e) Let $f : Z^+ \to Z^+$ where for all $x \in Z^+, f(x) = x+1$. What is the range of f? Is f one-to-one? Is it onto?

   Solution: The range of f is $\{2,3,4,...\}$. f is one-to-one, but not onto. $f^{-1}(1)$ is not defined.

(f) Let $f : Z^+ \to Z^+$ where for all $x \in Z^+, f(x) = max\{1, x-1\}$, the maximum of 1 and $x-1$. What is the range of f? Is f one-to-one? Is it onto?

Solution: The range of f is Z^+. f is not one-to-one since $f(1) = f(2) = 1$. f is onto.

(g) Prove or disprove: If functions $f : A \to B$ and $g : B \to C$ are onto, then $g \circ f$ is onto.

Solution: $g \circ f$ is onto. The domain of f is A and the codomain of g is C. It can be showed that for any $z \in C$, there exists $a \in A, b \in B$ such that $f(a) = b$, and $f(b) = z$.

2. Show that the function $f : R - \{3\} \to R - \{2\}$ defined by $f(x) = \frac{2x-3}{x-3}$ is a bijection, and find the inverse function.

   Solution: f is one-to-one:

   Suppose there exists distinct x_1 and x_2 such that $f(x_1) = f(x_2)$. In this case $\frac{2x_1-3}{x_1-3} = \frac{2x_2-3}{x_2-3}$. We can simplify this to show that $-3x_1 + 9 = -3x_2 + 9$. This is possible only when $x_1 = x_2$. Thus we have a contradiction to the fact that x_1 and x_2 are distinct. Therefore, $f(x_1)$ cannot be equal to $f(x_2)$ if x_1 and x_2 are distinct.

   f is onto

   Let $y (\neq 2)$ be an element of the codomain of f. We are interested in finding x in the domain such that $f(x) = y$, i.e. $\frac{2x-3}{x-3} = y$. Simplifying we get $x = \frac{3-3y}{2-y}$ which well defined for any $y \neq 2$. Therefore, f is onto.

   Computing f^{-1}

   We are interested in computing $f^{-1}(x) = y$, i.e. $f(y) = x$. Therefore, $y = \frac{3-3x}{2-x}$. Therefore $f^{-1}(x) = \frac{3-3x}{2-x}$. Note that the range of f^{-1} is $R - \{2\}$.

3. **Inclass test**
   Let $A = \{1,2,3,4,5\}$ and $B = \{1,2,3,4,5,6\}$. How many one-to-one functions $f : A \to B$ satisfy (a) $f(1) = 3$? (b) $f(1) = 3, f(2) = 6$.

   Solution: Number of one-to-one functions: 6.5.4.3.2 $= P(6,5)$.
   Number of one-to-one function with $f(1) = 3$ is 5.4.3.2
   Number of one-to-one function with $f(1) = 3$ and $f(2) = 6$ is 4.3.2.