1. Answer the following short questions.

(a) Give an example of a function \( f \) from integers to integers (\( f : \mathbb{Z} \to \mathbb{Z} \)) that is onto but not one-to-one.

**Solution:** The function \( f : \mathbb{Z} \to \mathbb{Z} \) defined by \( f(n) = \lfloor \frac{n}{2} \rfloor \) is onto but not one-to-one.

(b) Does the formula \( f(x) = \frac{1}{x+2} \) define a function \( f : \mathbb{R} \to \mathbb{R} \)? A function \( f : \mathbb{Z} \to \mathbb{R} \)?

**Solution:** When the domain of \( f \) is \( \mathbb{R} \), \( f \) is not a function since \( f(\sqrt{2}) \) is not defined. When the domain of \( f \) is \( \mathbb{N} \), \( f \) is a function.

(c) **Inclass**

Prove by cases that for any positive integer \( a \), \( \lfloor \frac{a}{2} \rfloor + \lceil \frac{a}{2} \rceil = a \).

**Solution:** The proof is “by cases”. When \( a \) is an even integer, i.e. \( a = 2k \), then \( \lfloor \frac{a}{2} \rfloor + \lceil \frac{a}{2} \rceil = k + k = 2k = a \).

When \( a = 2k + 1 \) (i.e. odd), then \( \lfloor \frac{a}{2} \rfloor + \lceil \frac{a}{2} \rceil = k + k + 1 = 2k + 1 = a \).

(d) If \( A = \{1, 2, 3, 4, 5\} \) and \( B = \{x, y, z\} \), determine the number of one-to-one, onto and constant functions one can design.

**Solution:** Number of one to one function = 0 (since \( |A| > |B| \)). Let \( A \) represent a set of guests and \( B \) represents a set of rooms.

- There are \( 3^5 \) ways to distribute the guests to rooms.
- Out of these, \( C(3, 1) \times 2^5 \) ways to distribute the guests with at least one room vacant.
- \( C(3, 2) \times 1^5 \) ways to distribute the guests with two vacant rooms.
- There are \( C(3, 1) \times 2^5 - C(3, 2) \times 1^5 \) ways to have a vacancy.
- Hence there are \( S(5, 3) = 3^8 - (C(3, 1) \times 2^5 - C(3, 2) \times 1^5) = 3^8 - C(3, 1) \times 2^5 + C(3, 2) \times 1^5 \) ways to occupy all rooms.

(e) Let \( f : \mathbb{Z}^+ \to \mathbb{Z}^+ \) where for all \( x \in \mathbb{Z}^+ \), \( f(x) = x + 1 \). What is the range of \( f \)? Is \( f \) one-to-one? Is it onto?

**Solution:** The range of \( f \) is \( \{2, 3, 4, \ldots\} \). \( f \) is one-to-one, but not onto. \( f^{-1}(1) \) is not defined.
(f) Let \( f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) where for all \( x \in \mathbb{Z}^+ \), \( f(x) = \max\{1, x - 1\} \), the maximum of 1 and \( x - 1 \). What is the range of \( f \)? Is \( f \) one-to-one? Is it onto?

**Solution:** The range of \( f \) is \( \mathbb{Z}^+ \). \( f \) is not one-to-one since \( f(1) = f(2) = 1 \). \( f \) is onto.

(g) Prove or disprove: If functions \( f : A \rightarrow B \) and \( g : B \rightarrow C \) are onto, then \( g \circ f \) is onto.

**Solution:** \( g \circ f \) is onto. The domain of \( f \) is \( A \) and the codomain of \( g \) is \( C \). It can be showed that for any \( z \in C \), there exists \( a \in A, b \in B \) such that \( f(a) = b \), and \( f(b) = z \).

2. Show that the function \( f : \mathbb{R} \setminus \{3\} \rightarrow \mathbb{R} \setminus \{2\} \) defined by \( f(x) = \frac{2x-3}{x-3} \) is a bijection, and find the inverse function.

**Solution:** \( f \) is one-to-one:

Suppose there exists distinct \( x_1 \) and \( x_2 \) such that \( f(x_1) = f(x_2) \). In this case \( \frac{2x_1-3}{x_1-3} = \frac{2x_2-3}{x_2-3} \). We can simplify this to show that \( -3x_1 + 9 = -3x_2 + 9 \). This is possible only when \( x_1 = x_2 \). Thus we have a contradiction to the fact that \( x_1 \) and \( x_2 \) are distinct. Therefore, \( f(x_1) \) cannot be equal to \( f(x_2) \) if \( x_1 \) and \( x_2 \) are distinct.

\( f \) is onto

Let \( y(\neq 2) \) be an element of the codomain of \( f \). We are interested in finding \( x \) in the domain such that \( f(x) = y \), i.e. \( \frac{2x-3}{x-3} = y \). Simplifying we get \( x = \frac{3-3y}{2-y} \) which well defined for any \( y \neq 2 \). Therefore, \( f \) is onto.

**Computing \( f^{-1} \)**

We are interested in computing \( f^{-1}(x) = y \), i.e. \( f(y) = x \). Therefore, \( y = \frac{3-3x}{2-x} \). Therefore \( f^{-1}(x) = \frac{3-3x}{2-x} \). Note that the range of \( f^{-1} \) is \( \mathbb{R} \setminus \{2\} \).

3. **Inclass test**

Let \( A = \{1, 2, 3, 4, 5\} \) and \( B = \{1, 2, 3, 4, 5, 6\} \). How many one-to-one functions \( f : A \rightarrow B \) satisfy (a) \( f(1) = 3 \)? (b) \( f(1) = 3, f(2) = 6 \).

**Solution:** Number of one-to-one functions: \( 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = P(6, 5) \).
Number of one-to-one function with \( f(1) = 3 \) is \( 5 \cdot 4 \cdot 3 \cdot 2 \)
Number of one-to-one function with \( f(1) = 3 \) and \( f(2) = 6 \) is \( 4 \cdot 3 \cdot 2 \).