MACM 101 D1 Section
Final Exam
December 11, 2018.

This test has two parts. Answer questions worth 50 points from Part A. From Part B, answer questions worth 30 points.

On the cover page of the exam booklet, please list all the questions you want to be marked in the same order as you have answered.

To get full credit, you must show all of your work.

Part A: Answer questions worth 50 points from this part.

1. (2 points) For each of the following sentences, determine whether an "inclusive or" or an "exclusive or " is intended. Justify tour answer. Note that this is a language problem, not a mathematical one.

   (a) Experience with C++ or Java is required.

   (b) Lunch includes soup or salad.

   (c) To enter the county you need a passport or a voter registration card.

   (d) Publish or perish.

2. (3 points) Show that if A, B, C are sets, then

$$A \oplus (B \oplus C) = (A \oplus B) \oplus C$$

3. (5 points) Suppose $x, y \in \mathbb{R}$. Prove that if $xy - x^2 + x^3 \geq x^2y^3 + 4$, then $x \geq 0$ or $y \leq 0$.

4. (5 points) Determine whether the following is a valid rule of inference:

$$\begin{array}{c} p \vee q \\ q \rightarrow r \\ p \rightarrow m \\ \neg m \\ \hline r \wedge (p \vee q) \end{array}$$

5. (5 points) How many positive integers less than 1000

   (a) are divisible by 7?
   (b) are divisible by 7 but not by 11?
   (c) are divisible by exactly one of 7 and 11?
   (d) are divisible by 12 but not by 14? (Note that 12 and 14 are not relatively prime numbers.)

6. (3 points) With n a positive integer, evaluate the sum by using the Binomial Expansion

$$\binom{n}{0} + 3\binom{n}{1} + 3^2\binom{n}{2} + \ldots + 3^k\binom{n}{k} + \ldots + 3^n\binom{n}{n}.$$

7. (5 points) Consider the following cosine function $f : A \to B$ where $f(x) = cos(x)$ (see Figure 1).

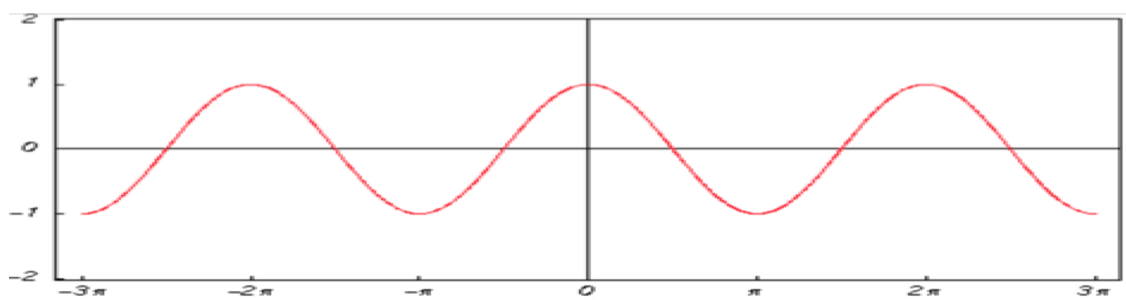

Figure 1: Cosine function

   (a) What is the domain and codomain of the function as shown.
   (b) Explain why the function is not invertible.
   (c) Specify a restricted domain over which the cosine function is invertible.

8. (7 points)

   (a) What is the main difference between regular mathematical induction and strong mathematical induction?
   (b) Suppose a_n is recursively defined as:

$$(Basis) \quad a_1 = 0$$

$$a_n = a_{n-1} + n(n-1) \quad \forall n \geq 1.$$

   Prove by induction that $a_n = \frac{n^3-n}{3}, \quad n \geq 1.$

9. (5 points) Consider a set $A = \{1, 2, 3, 4, 5, 6\}$.

   (a) Define an equivalence relation R on A which realizes $\{1, 3, 5\}$ and $\{2, 4, 6\}$ as the partition of A. Draw the directed graph representing R.

   (b) Define the relation R on A which is reflexive, symmetric, antisymmetric and transitive. Draw the directed graph representing R.

10. (5 points)

    (a) Prove or disprove: Among any 800 distinct integers chosen from the set $\{n : (n \in \mathbb{N}) \wedge (1 \leq n \leq 1600)\}$ at least two must be consecutive.

    (b) Prove or disprove: Among any 610 distinct integers chosen from the set $\{n : (n \in \mathbb{N}) \wedge (1 \leq n \leq 1200)\}$ at least two must be consecutive.

11. (5 points) What is the probability that a random relation from set $A = \{a, b, c, d\}$ to set $B = \{1, 2, 3, 4, ..., 8\}$ is a one-to-one function?

12. (5 points) Show that if $a, b \in \mathbb{Z}^+$ and p is prime, $p|ab \to p|a$ or $p|b$.

13. (5 points)

    (a) Find the prime factorization of 7!.

    (b) How many divisors of 7! are there?

    (c) Find the smallest perfect square that is divisible by 7!.

Part B: Answer questions worth 30 points from this part.

1. (5 points) For which positive integer n will the equations

$$x_1 + x_2 + x_3 + \ldots + x_{19} = n$$

$$y_1 + y_2 + y_3 + \ldots + y_{64} = n$$

   have the same number of positive integer solutions.

2. (5 points) Let Q denote the set of rational numbers. A rational number can be expressed as a ratio $\frac{a}{b}$ where $a, b \in \mathbb{Z}$, $b \neq 0$. Consider a function $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$. Of the two choices below show that one defines a function.

   (a) $f((\frac{a}{b}, \frac{c}{d})) = \frac{a+c}{b+d}$

   (b) $f((\frac{a}{b}, \frac{c}{d})) = \frac{ad+bc}{bd}$

3. (10 points) Let $\lfloor x \rfloor$ and $\lceil x \rceil$ respectively represent the "floor" and "ceiling" defined for any $x \in \mathbb{R}$ by

$$\lfloor x \rfloor = max\ \{n | (n \in \mathbb{Z})\ \wedge (n \leq x)\}$$

$$\lceil x \rceil = min\ \{n | (n \in \mathbb{Z})\ \wedge (n \geq x)\}$$

   (a) Prove that $\forall x \in \mathbb{R}$

$$\lfloor \tfrac{x}{2} \rfloor + \lfloor \tfrac{x+1}{2} \rfloor = \lfloor x \rfloor$$

   (b) It has been suggested that the following identity holds for the floor function:

$$\lfloor \tfrac{x}{2} \rfloor \times \lfloor \tfrac{x+1}{2} \rfloor = \lfloor \tfrac{x^2}{4} \rfloor.$$

   Is this an identity when $x \in \mathbb{R}$? Is this an identity when $x \in \mathbb{N}$?

4. (5 points)

   (a) Show that every integer greater than 11 is the sum of two composite integers. (All integer numbers greater than 1 are either composite or prime.)

   (b) Show that the preceding statement is "best possible", in the sense that 11 cannot be replaced by a smaller positive integer.

5. (5 points) Show that the two given sets have equal cardinality by describing a bijection (one-to-one and onto) from set A to set B. Describe your bijection with a formula.

   (a) $A = \{0, 1, 2, 3, \ldots \infty\}$ and $B = \{1, 2, 3, \ldots, \infty\}$

   (b) $A = \mathbb{Z}$ and $B = \{\ldots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \ldots\}$

6. (5 points) Let a, b be odd positive integers with $a > b$. Prove that $gcd(a, b) = gcd(\frac{a-b}{2}, b)$.

7. (5 points) The following questions involve modular arithmetic.

   (a) Determine the quotients and remainders of the following
   25 is divided by 15; -25 is divided by 15.

   (b) Compute $mod(11^{100}, 10)$.

   (c) Prove or disprove: For integers a and b, if $a \equiv b(mod\ 5)$, then $a^2 \equiv b^2(mod\ 5)$.

8. (5 points) Use the Euclidean algorithm - no other method will be acceptable - to determine integers m and n such that

$$gcd(341, 527) = 341m + 527n.$$