MACM 101 : Homework 5 (November 1, 2019)
Homework is due (in the tutorial class) Nov 13-14, 2019
Exercises on Properties of Integers (Chapter 4).

Practice Problems
1. Problems from the text: (page 219) 1, 3, 10, 11
2. Problems from the text: (page 230) 7, 8, 28
3. Problems from the text: (page 236) 3, 10, 14, 15, 19

Homework Problems
1. Consider \( n + 2 \) distinct points from the circumference of a circle. If consecutive points along the circle are joined by line segments creating a polygon with \( n + 2 \) sides then the sum of interior angle of the resulting polygon equals \( 180n \) degree.

   \[ \begin{array}{ccc} 
   1 & 2 & 3 \\
   \end{array} \]
   \[ \begin{array}{ccc} 
   1 & 2 & 3 \\
   \end{array} \]

2. Suppose that a sequence \( a_n \) \((n = 0, 1, 2, \ldots)\) is defined recursively by \( a_0 = 1, \ a_1 = 7, \ a_n = 4a_{n-1} - 4a_{n-2} \) \((n \geq 2)\). Prove by induction that \( a_n = (5n + 2)2^{n-1} \) for all \( n \geq 0 \).

3. Show that, for any positive integer \( n \), \( n \) lines “in general position” (i.e. no two of them are parallel, no three of them pass through the same point) in the plane \( \mathbb{R}^2 \) divide the plane into exactly \( \frac{n^2 + n + 2}{2} \) regions. (Hint: Use the fact that an \( n \)th line will cut all \( n - 1 \) lines, and thereby create \( n \) new regions.)

4. Give a recursive definition of the sequence \( \{a_n\}, n = 1, 2, 3, \ldots \), if
   (a) \( a_n = 4n \)
   (b) \( a_n = 4^n \)
   (c) \( a_n = 4 \)

5. Give a recursive definition for the set of all
   (a) positive even integers
   (b) positive odd integers
   (c) nonnegative even integers

6. Let \( n \in \mathbb{Z}^+ \) with \( n = r_0 + r_1 \times 6^1 + r_2 \times 6^2 + \ldots + r_k \times 6^k \). Prove that
   (a) \( 2|n \) if and only if \( 2|r_0 \).
   (b) \( 4|n \) if and only if \( 4|(r_0 + r_1 \times 6) \).
(c) $8|n$ if and only if $8|(r_0 + r_1 \times 6 + r_2 \times 6^2)$

7. (a) Determine the prime factorization of 374544.
   (b) Determine the number divisors of 374544 of types $a^i, i = 1, 2, 3, 4$ where $a$ is an integer.

8. (a) Use Euclidean algorithm to determine the greatest common divisor of the integers 243 and 198.
   (b) Use your computations above, determine two integers, $x$ and $y$, such that $\gcd(243, 198) = 243x + 198y$.
   (c) Determine a value of $c$ such that $c = 243a + 198b$ where $a, b, c \in \mathbb{Z}^+$.

9. Determine the value of $c \in \mathbb{Z}^+, 30 < c < 39$, for which the Diophantine equation $243a + 198b = c$ has no solution. Determine the solutions of the remaining values of $c$. 