Practice Problems

1. Problems from the text: (page 219) 1, 3, 10, 11
2. Problems from the text: (page 230) 7, 8, 28
3. Problems from the text: (page 236) 3, 10, 14, 15, 19

Homework Problems

1. Consider \( n + 2 \) distinct points from the circumference of a circle. If consecutive points along the circle are joined by line segments creating a polygon with \( n + 2 \) sides then the sum of interior angle of the resulting polygon equals \( 180n \) degree.

Solution: Let \( S(n) \) be the open statement the sum of the interior angles of \( n + 2 \) sided polygon is \( 180n \) degrees. We want to show that \( \forall n \geq 1, \ S(n) \) is true. We will use the weak induction to prove the claim.

Basis: \( S(1) \) is true since the sum of the interior angles of a triangle is 180, i.e. \( 180 \times 1 \) degrees.

Inductive hypothesis: Suppose \( S(k) \) is true for an arbitrary \( k \geq 1 \), i.e. the sum of the interior angles of any \( k + 2 \) vertices polygon is \( 180 \times k \). This means that the interior of the polygon can be triangulated with \( k \) triangles.

Showing \( S(k+1) \) is true: Consider an arbitrary convex polygon with \( (k+1)+2 \) vertices, say \( v_1, v_2, \ldots, v_k+2, v_{k+3} \), on the boundary of the circle. We select an arbitrary vertex \( v_i \). After it is removed the remaining \( k + 2 \) vertices form a polygon \( P \) whose total interior angles is \( 180 \times k \) by the induction hypothesis. We now add \( v_i \) to the triangulated \( P \) and the resulting polygon is triangulated with \( k + 1 \) triangles. In the figure, we can add the triangle 3 to the triangulated polygon with 4 vertices. This implies that the sum of the interior angles of a polygon with \( k + 3 \) vertices is \( 180 \times (k + 1) \). Therefore, \( S(1) \land S(k) \rightarrow S(k+1) \). By the principle of weak mathematical induction, \( S(n) \) is true for all \( n \geq 1 \).

2. Suppose that a sequence \( a_n \ (n = 0, 1, 2, \ldots) \) is defined recursively by \( a_0 = 1, 
\quad a_1 = 7, \ a_n = 4a_{n-1} - 4a_{n-2} \ (n \geq 2) \). Prove by induction that \( a_n = (5n + 2)2^{n-1} \) for all \( n \geq 0 \).

Solution: Let \( S(n) \) be the open statement \( a_n = (5n + 2)2^{n-1} \). We need to show that \( \forall n \geq 0 \ S(n) \) is true. Clearly, \( S(0) \) and \( S(1) \) are true. We will use the strong induction to prove our claim. We show that \( S(0) \land S(1) \land S(k-2) \land S(k-1) \rightarrow S(k+1) \).
7. (a) Determine the prime factorization of 374544.  
\[ a_{k+1} = 4a_k - 4a_{k-1} \text{ from the definition} \]
\[ = 2^2(5k+2)2^{k-1} - 2^2(5(k-1)+2)2^{k-2} \]
\[ = (5k+2)2^{k+1} - (5(k-1)+2)2^k \]
\[ = 2^k(10k+4 - 5k+5 - 2) \]
\[ = 2^k(5k + 1) + 2 \]

We thus show that \( S(k-1) \land S(k) \rightarrow S(k+1) \). Therefore, by the principle of strong induction, we can claim that \( \forall n \geq 0 S(n) \) is true.

3. Show that, for any positive integer \( n \), \( n \) lines “in general position” (i.e., no two of them are parallel, no three of them pass through the same point) in the plane \( \mathbb{R}^2 \) divide the plane into exactly \( \frac{n^2 + n + 2}{2} \) regions. (Hint: Use the fact that an \( n \)th line will cut all \( n - 1 \) lines, and thereby create \( n \) new regions.)  
**Solution:** Let \( a_n \) be the number of regions created by \( n \) lines. The recursive formulation of \( n \) is  
\[ a_1 = 2, \text{ and } a_n = a_{n-1} + n, n \geq 2. \]
We have to show that \( a_n = \frac{n^2 + n + 2}{2} \). The proof is very similar to that of the above two questions.

4. Give a recursive definition of the sequence \( \{ a_n \}, n = 1, 2, 3, \ldots \), if  
(a) \( a_n = 4n \)  
(b) \( a_n = 4^n \)  
(c) \( a_n = 4 \)  
**Solution:** \( a_1 = 4; a_n = a_{n-1} + 4, n \geq 2 \)

5. Give a recursive definition for the set of all  
(a) positive even integers  
(b) positive odd integers  
(c) nonnegative even integers  
**Solution:** \( 2 \in S, \text{ if } x \in S, x + 2 \in S \)

6. Let \( n \in \mathbb{Z}^+ \) with \( n = r_0 + r_1 \times 6^1 + r_2 \times 6^2 + \ldots + r_k \times 6^k \). Prove that  
(a) \( 2|n \) if and only if \( 2|r_0 \).
(b) \( 4|n \) if and only if \( 4|(r_0 + r_1 \times 6) \).
(c) \( 8|n \) if and only if \( 8|(r_0 + r_1 \times 6 + r_2 \times 6^2) \)
**Solution:** Suppose \( p(n) : 8|n \) and \( q(n) : 8|r_0 + r_1 \times 6^1 + r_2 \times 6^2 \). We want to show that \( \forall n \ p(n) \leftrightarrow q(n) \). We first show that \( \forall n \ p(n) \rightarrow q(n) \). The equivalent contrapositive statement we need to prove is \( \forall n \neg q(n) \rightarrow \neg p(n) \). Since \( \neg q(n) \) is true, \( r_0 + r_1 \times 6^1 + r_2 \times 6^2 = 8.T + i \) where \( T \) and \( i \) are integers and \( 1 \leq i \leq 7 \). Now we can write \( n = 8.T + i + 8.T' \), since 8 divides \( r_3 \times 6^3 + r_4 \times 6^4 + \ldots + r_k \times 6^k \). Thus \( n = 8(T + T') + i \). This shows that \( 8 \not| n \).
We can easily prove the second part: \( \forall n \ q(n) \rightarrow p(n) \).

7. (a) Determine the prime factorization of 374544.  
**Solution:** \( 4374544 = 2^4 \cdot 3^4 \cdot 17^2 \).
(b) Determine the number divisors of 374544 of types \(a^i, i = 1, 2, 3, 4\) where \(a\) is an integer.  
**Solution:** Suppose we are interested in finding the number of divisors which are perfectly squares. For example, \(2^2, 2^4, 3^2, 3^4, 2^23^2\), etc. The number of such divisors is \((\left\lfloor \frac{4}{2} \right\rfloor + 1) \times (\left\lfloor \frac{4}{2} \right\rfloor + 1)\). Note here that \(1^2\) is considered as a divisor which is a perfect square.

8. (a) Use Euclidean algorithm to determine the greatest common divisor of the integers 243 and 198.  
**Solution:** We compute the following:\n\begin{align*}  
243 &= 1.198 + 45; 198 = 4.45 + 18; 45 &= 2.18 + 9; 18 &= 2.9 + 0. \text{ Therefore,} \end{align*}
\[\text{gcd}(243, 198) = 9.\]

(b) Use your computations above, determine two integers, \(x\) and \(y\), such that \(\text{gcd}(243, 198) = 243x + 198y\).  

Thus we have \(x = 9\) and \(y = -11.\)

(c) Determine a value of \(c\) such that \(c = 243a + 198b\) where \(a, b, c \in \mathbb{Z}^+\).  
**Solution:** We can arbitrarily set \(a = 1\) and \(b = 1\), and we get \(c = 243.1 + 198.1 = 441.\) Note that \(a, b, c \in \mathbb{Z}^+\).

9. Determine the value of \(c \in \mathbb{Z}^+, 30 < c < 39\), for which the Diophantine equation \(243a + 198b = c\) has no solution. Determine the solutions of the remaining values of \(c\).  
**Solution:** We know \(\text{gcd}(243, 198) = 9\) and we can write 9 = 9 \times 243 – 11 \times 198. Since 9 divides only 36 in the range \([31, 38]\), \(243a + 198b = c\) have no solution when \(c = 31, 32, 33, 34, 35, 37, 38\). When \(c = 36\), we can write 36 = 243(9 \times 4) + 198(–11 \times 4). We can rewrite 36 = 243(9 \times 4 – 198k) + 198(–11 \times 4 + 243k) for any \(k\). For every integer values of \(k\), we obtain a different integral solution to the equation \(243a + 198b = c\).