CMPT 307 Homework 2
September 23, 2019
Practice Problems for Quiz 1.
Homework is due on Monday, September 30, 2019.

1. Practice Problems (Chapter 1) 1.1, 1.3, 1.5, 1.7, 1.10, 1.11, 1.12, 1.15, 1.17, 1.19, 1.20, 1.22, 1.25, 1.26, 1.31, 1.39.
2. Consider the prime factorization of $48620250 (= n) = 2^1 \times 3^2 \times 5^3 \times 7^4 \times 11^1$.
   (a) Determine the number of divisors of n.
   (b) Determine the sum of the divisors of n.

Homework Problems

1. Programming Contest Problems Just describe how you plan to solve these problems. A trivial brute force like approach is not a good idea.

   Divisors

   You can find the problem statement from `www.cs.sfu.ca/~binay/2019/cmpt307/Divisors.pdf`.

   Solution Sketch The problem can be restated as follows: Given a range of numbers between 1 and 1,000,000,000 and the range of the numbers is no greater than 10,000, find the number in the range with the greatest number of divisors as well as the number of divisors it has. We will use the following information.

   (a) Any number n can be expressed as a product of prime numbers (Question 2 of this homework). Let $n = p_1^{n_1}.p_2^{n_2} \dots p_k^{n_k}$ where $p_i, i = 1, 2, \dots, k$ are distinct primes and $n_i, i = 1, 2, \dots, k$ are integers.
   (b) We find the factorization by trying each prime of value no more than $\sqrt{n}$.
   (c) We can compute all the primes in the range $[1, 10^9]$ once and use them during all the test cases.
   (d) We recognise the fact that if $n = p_1^{n_1}.p_2^{n_2} \dots p_k^{n_k}$, the number of divisors of n is $(n_1 + 1)(n_2 + 1) \dots (n_k + 1)$.

   The above observations will be enough to avoid the time limit violation.

   Marbles

   You can find the problem statement from `www.cs.sfu.ca/~binay/2019/cmpt307/Marbles.pdf`.

Solution: For the sake of keeping it simple, let us consider a concrete example. Suppose there are $n = 43$ marbles. Suppose Type one and two boxes can hold 4 and 3 marbles, respectively. We are interested in finding integers α and β, both ≥ 0, such that $n = \alpha \times 4 + \beta \times 3$.

We first notice that $gcd(4, 3) = 1$. Since $gcd(4, 3)$ divides n, therefore, the solution as required may exist. We also know that $gcd(4, 3)$ can be expressed as $1 = 4.x + 3.y$ where $x = 1$ and $y = -1$. We can write $43 = 43.4.x + 43.3.y$. We can rewrite this as $43 = 4(43.x - 3.t) + 3(43.y + 4.t)$ any t. We just subtracted 4.3.t and then added the same amount. Now our objective is to select an integer value for t, if it exists, such that $43.x - 3t \geq 0$ and $43.y + 4t \geq 0$. Note that $43.x - 3t \geq 0$ implies that $t \leq \frac{43.x}{3}$ i.e. $t \leq \frac{43}{3}$. Similarly $43.y + 4t \geq 0$, i.e. $43.(-1) + 4t \geq 0$ implies that $t \geq \frac{43}{4}$. We note that when $t = 11, 12, 13$ and 14 both $43.x - 3t \geq 0$ and $43.y + 4t \geq 0$. Therefore, we have the following solution (when $t = 11$): $43 = 4(43.1 - 33) + 3(43. - 1 + 44) = 4\alpha + 3\beta, \alpha = 10, \beta = 1$. We can also have other solutions by plugging $t = 12, 13$ or 14.

2. Let n be a positive integer and let S be a subset of $n + 1$ elements of the set $\{1, 2, 3, \ldots, 2n\}$.Show that

   (a) There exist two elements of S that are relatively prime, and

   (b) There exist two elements of S, one of which divides the other.

   Solution
   (a) There must exist two elements a and b of S which are consecutive, and therefore, $gcd(a, b) = 1$.
   (b) Find the greatest odd factor of each element of S. These factors are elements of the set $\{1, 3, 5, \ldots, 2n - 1\}$ (containingSince the size of S is $n + 1$, there are two elements of S with the same greatest odd factors. Therefore they differ (multiplication-wise) by a power of 2, thus one dividing the other.