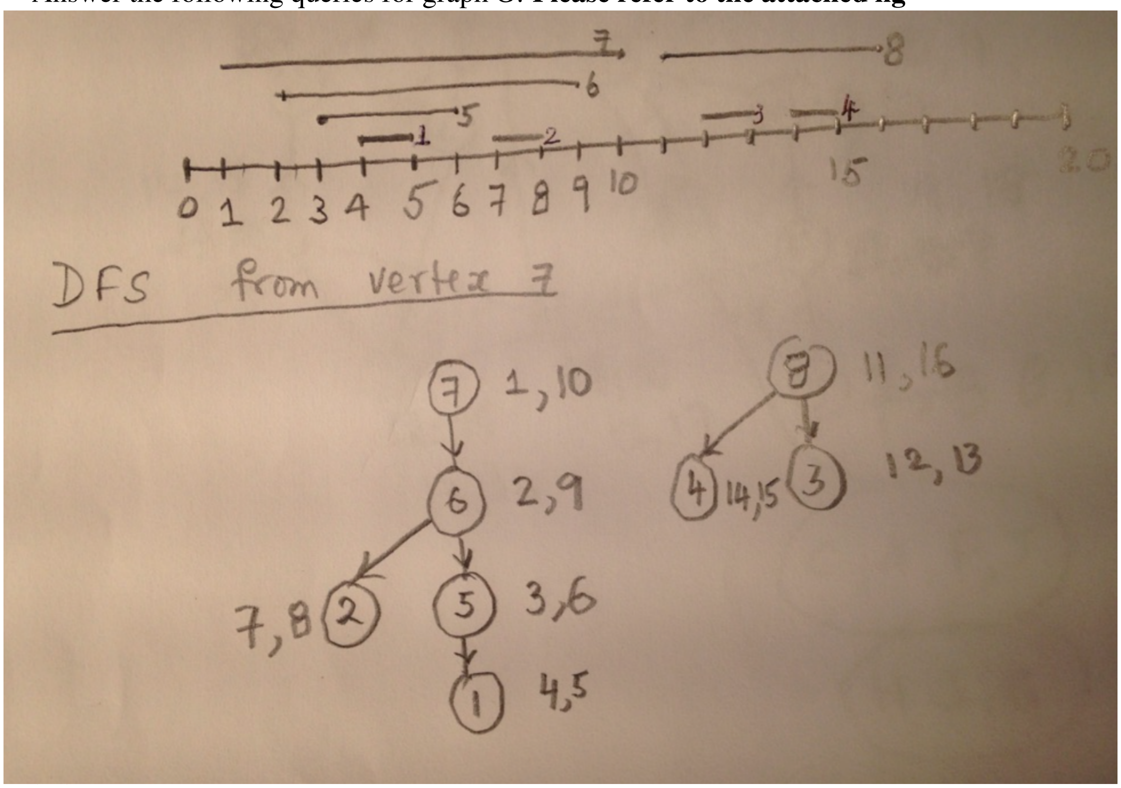
CMPT 307 : Quiz 2 (Total Marks: 40) October 23, 2019

Answer questions 1 and 2, and any 20 points from the rest.

1. (10 points) Suppose that we are given a set of depth first intervals of the nodes of a graph G as follows:

$$v[1] : [4,5];\ \ v[2] : [7,8];\ \ v[3] : [12,13];\ \ v[4] : [14,15];\ \ v[5] : [3,6];$$
$$v[6] : [2,9];\ \ v[7] : [1,10];\ \ v[8] : [11,16].$$

Answer the following queries for graph G. **Please refer to the attached fig-**



(a) What are the descendant and ancestor nodes of $v[6]$ in G?
Ans: The ancestor node: 7; descendant nodes: $1, 2, 5$

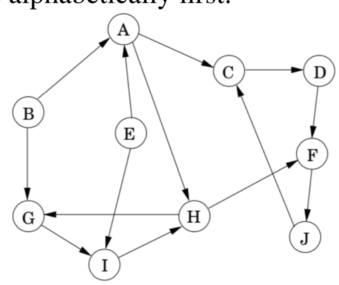
(b) How many components are there in G? **Ans: There are two connected components.**

(c) Identify a pair of nodes in a connected component of G which are not related (i.e. one is neither a descendant nor an ancestor of the other). **Ans: nodes 1 and 2. Nodes 3 and 4; and 2 and 5 are also not related.**

(d) Construct the depth first tree of G which realizes the dfs intervals as given. **Ans: Please see the attached figure.**

(e) Remove one node from G such that the number of connected components remains the same. (Note that *G* may have many edges which we are not aware of.) **Ans: Any leaf node of a tree of size at least two in the DFS forest can be removed without increasing/decreasing the number of components. We can, therefore, remove** $1, 2, 3$ *or* 4**.**

2. (10 points) Run the strongly connected components algorithm on the following directed graph *G*. Whenever there is a choice of vertices to explore, always pick the one that is alphabetically first.



Answer the following questions.

(a) In what order are the strongly connected components (SCCs) found? **Ans: The strongly connected components determined in order are: E, B, A; HGI, CDFJ; The algorithm I have used to find the SCCs is:**

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Step 1: Perform DFS on G and compute [pre(v),post(v)]
        interval for all v.
Step 2: Order the vertices in decreasing post(*) values.
Step 3: Compute the reverse graph G^R of G.
Step 4: Run DFS on G and during the DFS, process the
        vertices in decreasing order of their post
        numbers from step1.
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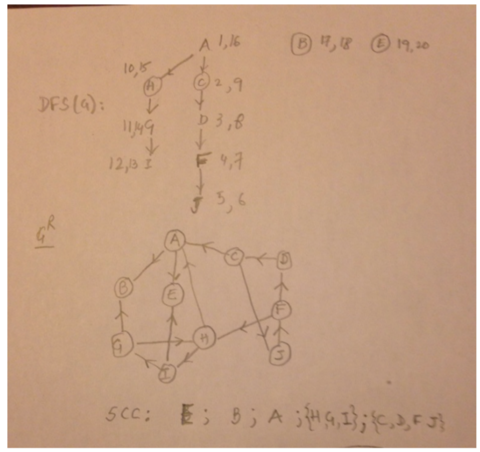
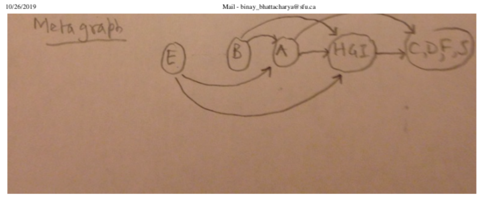
(b) Which are source SCCs and which are sink SCCs?
Ans: The source SCCs: E and B; sink SCC: CDFJ.

(c) Draw the "metagraph".
Ans: See the following figure.





(d) What is the minimum number of edges you must add to this graph to make it strongly connected?
Ans: Two directed edges (one directed from E to B, and the other from C to E), when added, will make the graph G with no source or sink node.

3. (10 points) It is easy to see that for any graph G, both DFS and BFS will take almost the same amount of time. However the space requirements may be significantly different.

   (a) Give an example of an n-vertex graph for which the height of the recursion tree of DFS from a particular vertex v is $n-1$ whereas the queue of BFS will have at most one vertex at any given time if BFS is started from the same vertex.
   Ans: When the graph is a path as follows,

```
  o-----o------o-------o--------o------o
v(1)  v(2)   v(3)                     v(n)
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   the height of the recursion tree of DFS starting from $v(1)$ will be $n-1$. The queue size of BFS will be one.

(b) Give an example of an n vertex graph for which the queue of BFS will have $n-1$ vertices at one time whereas the height of the recursion tree of DFS is at most one. Both searches are started from the same vertex.
Ans: The graph has the following edges:

$$(v(1),v(2)),(v(1),v(3)),(v(1),v(4)),\ldots,(v(1),v(n))$$

and both the searches start from $v(1)$.

4. (10 points)

(a) Suppose G is a connected undirected graph. An edge e whose removal disconnects the graph is called a bridge. Must every bridge e be an edge in a depth-first search tree of G, or can e be a back edge? Either give a proof or a counterexample.
Ans: The removal of a bridge $e=(u,v)$ disconnects the graph. Let C_u and C_v be the two connected components containing u and v respectively. The edge (u,v) connects C_u and C_v. Without any loss of generality suppose DFS of G enters C_u first. The search will enter C_v from u. All the vertices of C_v will be descendant nodes of v. There will be no back edge from C_v to C_u. This forces all DFS of G to use the edge (u,v) as a tree edge.

(b) Using a DFS on G, can you identify a bridge edge of G, if it exists.
Ans: From the above discussion we notice that when we are performing DFS and when we are backing up along a tree edge (u,v) where $pre(u) < pre(v)$, we check if there is any back edge from v or any descendant of v to an ancestor of v. The ancestor nodes of v are u and all the ancestor nodes of u. (u,v) is not a bridge if there is any such back edge, otherwise it is a bridge edge. This can be determined during the DFS process. It is possible to identify all the bridge edges of G in $O(|V|+|E|)$ time. Problem 3.31 of the text deals with this problem.

5. (10 points) A mother vertex in a directed graph $G=(V,E)$ is a vertex v such that all other vertices G can be reached by a directed path from v.

(a) Give an $O(n+m)$ algorithm to test whether a given vertex v is a mother of G, where $n=|V|$ and $m=|E|$.
Ans: We start DFS from v and check if the DFS is a tree with $|V|-1$ tree edges.

(b) Give an $O(n+m)$ algorithm to test whether graph G contains a mother vertex.

Ans: Identify a source SCC vertex in the metagraph. Pick any vertex in the selected SCC component. Then check if it is mother vertex.

6. (10 points **Bonus**) **Finding the Topological Sort of a Directed Acyclic Graph**

   Let G=(V,A) be a directed acyclic graph that has an edge between every pair of vertices and whose vertices are labeled $1, 2, \ldots, n$, where $n = |V|$. To determine the direction of an edge between two vertices in V, you are only allowed to ask a query. A query consists of two specified vertices u and v and is answered with:

   - "from u to v" if (u, v) is in A, or
   - "from v to u" if (v, u) is in A.

   Determine the number of queries required in the worst case, as a function of n, to find a topological sort of G.

   Ans: Our directed graph $G = (V, A)$ is a DAG and there is a directed edge between every two vertices, i.e. for any u and v, of V, either (u, v) or (v, u), but not both, is an element of A. Since G is a DAG, we can topologically sort the vertices of G, i.e. we can order the vertices on a line from left to right such that for any i, j, $i < j$ in the ordered list, the arc (i, j) is an element of A. We can treat the topologically ordering the vertices problem as a comparison-based sorting problem since for any two vertices u and v, u is to the left of v (i.e. u is smaller than v) in the sorted order if there is an arc from u to v. If $(v, u) \in A$, v lies to the left of u in the sorted order.

   We know that we can sort n elements using $O(nlogn)$ comparisons (merge-sort, heapsort). Therefore we require $O(nlogn)$ queries to topologically order the vertices of G.