MACM 101 : Quiz 2.1 (Wednesday/Thursday) (Full Marks 50) Time: 40 minutes

1. (5) Decide if the following are statements. If yes then state if it is True or False.

   (a) sets $\mathbb{Z}$ and $\mathbb{N}$
   It's not a statement.

   (b) $\mathbb{N} \notin \mathscr{P}(\mathbb{N})$
   It's a statement and it is true.

   (c) 4+x=5
   It's not a statement, because the truth value depends on the variable x.

   (d) 1+1=3 if and only if 2+2=3
   It's a statement and it's true.

   (e) If it's raining then it's raining
   It's a statement and it's true.

   (f) If 1 = 0 then 3=4
   It's a statement and it's true.

   (g) $100 \leq 2^n$
   It's not a statement, because the truth value depends on the variable n.

   (h) $p \vee \neg p$ is a tautalogy.
   It's a statement and it's true.

2. (5) Express each statement in the form "If P, then Q" :

   (a) For a function to be continuous, it is sufficient that it is differentiable
   If a function is differentiable, then it's continuous.

   (b) A geometric series with ratio r converges if $|\, r \,| < 1$.
   If a geometric series ratio $|\, r \,| < 1$, then the geometric series converges.

   (c) It is not hot whenever it is sunny
   If it is sunny, then it is not hot.

   (d) You get a good grade only if you study
   If you want to get a good grade, then you should study.

   (e) Studying is sufficient for passing
   If you are studying, then you will pass.

   (f) The team wins if the quarterback can pass
   If the quarterback can pass, then the team wins.

   (g) You need to be registered in order to check out library books.
   If you are registered, then you can check out library books.

   (h) The beach erodes whenever there is a storm
   If there is a storm, then the beach erodes.

   (i) It is necessary to have a valid password to log on to the server
   If you want to log on the server, then you should have a valid password.

3. (5) Write down the truth tables for the following statements:

(a) $[\neg p \wedge (p \vee q)] \rightarrow q$

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

(b) $(p \oplus q) \wedge (p \oplus \neg q)$

p	q	$\neg q$	$p \oplus q$	$p \oplus \neg q$	$(p \oplus q) \wedge (p \oplus \neg q)$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	F

(c) $((p \rightarrow q) \wedge \neg p) \rightarrow \neg q$

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$((p \rightarrow q) \wedge \neg p)$	$((p \rightarrow q) \wedge \neg p) \rightarrow \neg q$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

(d) $((p \rightarrow \neg q) \wedge \neg p)$

p	q	$\neg p$	$\neg q$	$p \rightarrow \neg q$	$(p \rightarrow \neg q) \wedge \neg p$
T	T	F	F	F	F
T	F	F	T	T	F
F	T	T	F	T	T
F	F	T	T	T	T

(e) $(q \vee r) \Leftrightarrow (r \wedge q)$

q	r	$q \vee r$	$r \wedge q$	$(q \vee r) \Leftrightarrow (r \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

(f) $(p \oplus q) \rightarrow (p \wedge q)$

p	q	$p \wedge q$	$p \oplus q$	$(p \oplus q) \rightarrow (p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	F
F	F	F	F	T

4. (10) Using the laws of logic, decide whether or not the following pair is logically equivalent.

(a) $P \Rightarrow Q$ and $\neg(P \vee \neg Q)$

$$\text{left side} = \neg P \vee Q \tag{1}$$
$$\tag{2}$$

$$\text{right side} = \neg P \wedge \neg(\neg Q) \tag{3}$$
$$= \neg P \wedge Q \tag{4}$$

Thus, the left side and the right side are not logically equivalent.

(b) $P \wedge Q \Rightarrow R$ and $P \Rightarrow (Q \Rightarrow R)$.

$$\text{left side} = \neg(P \wedge Q) \vee R \tag{5}$$
$$= \neg P \vee \neg Q \vee R \tag{6}$$

$$\text{right side} = P \Rightarrow (\neg Q \vee R) \tag{7}$$
$$= \neg P \vee (\neg Q \vee R) \tag{8}$$

Thus, the left side and the right side are logically equivalent.

(c) $\overline{\overline{p \vee \overline{(q \wedge r)}} \vee \overline{p \wedge \overline{q}}}$ and $p \wedge \overline{q}$. (Note that we have used $\overline{*}$ to indicate $\neg *$).

$$\text{left side} = \overline{\overline{p \vee \overline{(q \wedge r)}}} \wedge \overline{\overline{p \wedge \overline{q}}} \tag{9}$$
$$= \big(p \vee (q \wedge r)\big) \wedge (p \wedge \overline{q}) \tag{10}$$
$$= \big(p \wedge (p \wedge \overline{q})) \vee ((q \wedge r) \wedge (p \wedge \overline{q}))(distributive\ \ law) \tag{11}$$
$$= \big(p \wedge \overline{q}\big) \tag{12}$$

The two terms in eqn. (11) are connected by a $\vee$. The second term is false since it involves $q \wedge \overline{q}$.

(d) $(p \wedge q) \vee (q \wedge r) \vee (q \vee r)$ and $q \wedge (p \vee r)$.

$$\text{left side} = (p \wedge q) \vee \big((q \wedge r) \vee q \vee r\big) \tag{13}$$
$$= (p \wedge q) \vee \big((q \wedge r) \vee q\big) \vee r\big) \tag{14}$$
$$= (p \wedge q) \vee (q \vee r) \tag{15}$$
$$= (p \wedge q) \vee q \vee r \tag{16}$$
$$= q \vee r \tag{17}$$

Since $q \wedge (p \vee r) \subseteq (q \vee r)$, the left side and the right side are not logically equivalent.

5. (10) Express the statements in predicate/Symbolic logic:

   (a) Every integer greater than 2 is the sum of two primes.
   Let $P(x)$: x is the sum of two primes.
   $\forall x \in \mathbb{Z}, x > 2, p(x)$.

(b) For every prime number p there is another prime number q with q > p.
Let $P(p,q)$: q > p
U: the universal set of prime numbers
$\forall p \in U, \exists q \in U, P(p,q)$.

(c) Every user has access to electronic mail box
Let $p(x)$: x has access to electronic mail box.
$\forall x, p(x)$

(d) No rabbit knows Calculus
Let U be the universal set of rabbits
Let $p(x)$: x knows Calculus, $x \in U$
$\forall x \in U, \neg p(x)$

(e) There is a student in the class who has taken more than 21 credits and has received all A's
Let U be the universal set of students in the class
Let $p(x)$: x has taken more than 21 credits, $x \in U$.
Let $q(x)$: x has received all A's, $x \in U$.
$\exists x \in U, p(x) \wedge q(x)$

(f) Every course is taken by at least one student
Let U be the universal set of courses
Let $p(x)$: x is taken by at least one student, $x \in U$.
$\forall x \in U, p(x)$

(g) If a person is friendly , then that person is not angry.
Let U be the universal set of persons
Let $p(x)$: x is friendly, $x \in U$.
Let $q(x)$: x is not angry, $x \in U$.
$\forall x \in U, p(x) \rightarrow q(x)$

6. (5) Express the statement in predicate/symbolic logic. Form the negation of the statement.

(a) Either $x = 0$ or $y = 0$.
Let $p(x)$: $x = 0$
$p(x) \vee p(y)$
The negation of the statement is as following:
$\neg(p(x) \vee p(y))$ which is equivalent to $\neg\, p(x) \wedge \neg p(y)$, or: Both x and y are non-zero.

(b) Integers a and b are both nonnegative.
Let $p(x)$: x is nonnegative, $x \in Z$
$p(a) \wedge p(b)$
The negation of the statement is as following:
$\neg(p(a) \wedge p(b)) \Leftrightarrow \neg p(a) \vee \neg p(b)$

(c) If f is a polynomial and its degree is greater than 1, the f' is not constant.
Let U : universe of polynomials; $d(f)$: degree of f is greater than 1; $r(f)$: the derivative of f is continuous.

$\forall f \;\; d(f) \Rightarrow r(f)$
The negation of the statement is
$\neg(\forall f \;\; d(f) \Rightarrow r(f)) \Leftrightarrow \exists f, \neg(\neg d(f) \vee r(f)) \Leftrightarrow \exists f, \; (d(f) \wedge \neg r(f))$.

(d) There exists a real number r such that $r^2 = 2$.
let $p(x)$: $x^2 = 2$
$\exists r \in \mathbb{R}, p(r)$
The negation of the statement is as following:
$\neg\big(\exists r \in \mathbb{R}, p(r)\big) \Leftrightarrow \forall r \in \mathbb{R}, \neg p(r)$

7. (10) Find a proposition with the truth table

p	q	?
T	T	F
T	F	T
F	T	T
F	F	T

$(p \wedge (\neg q)) \vee ((\neg p) \wedge q) \vee ((\neg p) \wedge (\neg q))$

8. (Bonus)(10) Consider the statement P : Given any subset X of $\mathbb{R}$, there exists a subset Y of $\mathbb{R}$ for which $X \cap Y = \Phi$ and $X \cup Y = \mathbb{R}$.

   (a) Express the statement in predicate/Symbolic logic. Is the statement P true or false?
   $\forall X \subseteq \mathbb{R}, \exists Y \subseteq \mathbb{R}, (X \cap Y = \Phi) \wedge (X \cup Y = \mathbb{R})$
   It is True. Given any subset $\mathbb{R}$, let $Y = \overline{X}$ where $\overline{X}$ is the complement of X. Now $\overline{X}$ is another subset of $\mathbb{R}$. Now $X \cap Y$ is empty and $X \cup Y$ is $\mathbb{R}$.

   (b) Form the negation $\neg P$ symbolically. Write your answer as an English statement.

   $\neg\big(\forall X \subseteq \mathbb{R}, \exists Y \subseteq \mathbb{R}, X \cap Y = \Phi \wedge X \cup Y = \mathbb{R}\big)$
   $\Leftrightarrow \exists X \subseteq \mathbb{R}, \forall Y \subseteq \mathbb{R}, X \cap Y \neq \Phi \vee X \cup Y \neq \mathbb{R}\big)$
   There exists a subset X of $\mathbb{R}$, such that for every subset Y of $\mathbb{R}$, the intersection of X and Y is not the empty set or the union of X and Y is not the rational number set $\mathbb{R}$.

9. (Bonus)(10) Consider the statement P : Given any nonzero $x \in \mathbb{R}$, there exists an element $y \in \mathbb{R}$ for which $xy = 1$.

   (a) Express the statement in predicate/Symbolic logic. Is the statement P true or false?
   Let $p(x,y)$: $xy = 1$.
   $\forall x \in \mathbb{R}(x \neq 0 \Rightarrow \exists y \in \mathbb{R}, p(x,y))$
   True

   (b) Form the negation $\neg P$ symbolically. Write your answer as an English statement.
   $\neg \forall x \in \mathbb{R}(x \neq 0 \Rightarrow \exists y \in \mathbb{R}, p(x,y))$

   $\Leftrightarrow \exists x, \neg(\neg(x \neq 0) \vee \exists y, p(x,y))$

$\Leftrightarrow \exists x, (x \neq 0) \wedge (\forall y\ , \neg p(x,y))$
There exists a nonzero real x for which $xy \neq 1$ for every real number y.