Solution Set of Induction

2. For every integer $n \in \mathbb{N}$, it follows that

$$1^2+2^2+3^2+4^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

Proof. Suppose $S(n): 1^2+2^2+3^2+4^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$, for $n \geq 1$. We need to show that $\forall n \geq 1, S(n)$ is true.
Basic step: when $n=1$, then $1^2=\frac{1*(1+1)(2*1+1)}{6}=1$. Therefore, $S(1)$ is true.
Inductive step: Suppose that when $S(k): 1^2+2^2+3^2+4^2+\cdots+k^2=\frac{k(k+1)(2k+1)}{6}$ is true.
We want to prove that $S(k+1): 1^2+2^2+3^2+4^2+\cdots+k^2+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}$ is true.

$$\begin{aligned} l.h.s=1^2+2^2+3^2+4^2+\cdots+k^2+(k+1)^2 &= \frac{k(k+1)(2k+1)}{6}+(k+1)^2 \\ &= \frac{(k+1)(k(2k+1)+6(k+1))}{6} \\ &= \frac{(k+1)(2k^2+k+6k+6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= r.h.s \end{aligned} \tag{1}$$

Thus, $S(k+1): 1^2+2^2+3^2+4^2+\cdots+k^2+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}$ is true.
Thus by the principle of mathematical induction $\forall n \geq 1, S(n)$ is true. □

4. If $n \in \mathbb{N}$, then $1*2+2*3+3*4+4*5+\cdots+n(n+1)=\frac{n(n+1)(n+2)}{3}$.

Proof. Suppose $S(n): 1*2+2*3+3*4+4*5+\cdots+n(n+1)=\frac{n(n+1)(n+2)}{3}$, for $n \geq 1$. We need to show that $\forall n \geq 1, S(n)$ is true.
Basic step: when $n=1$, then $1*2=2=\frac{1*(1+1)*(1+2)}{3}=2$. Therefore, $S(1)$ is true.
Inductive step: suppose that when $S(k): 1*2+2*3+3*4+4*5+\cdots+k(k+1)=\frac{k(k+1)(k+2)}{3}$ is true. Thus, we want to prove that when $S(k+1): 1*2+2*3+3*4+4*5+\cdots+k(k+1)+(k+1)(k+2)=\frac{(k+1)(k+2)(k+3)}{3}$ is true.

$$\begin{aligned} & l.h.s.=1*2+2*3+3*4+4*5+\cdots+k(k+1)+(k+1)(k+2) \\ & =\frac{k(k+1)(k+2)}{3}+(k+1)(k+2) \\ & =\frac{(k+1)(k(k+2)+3(k+2))}{3} \\ & =\frac{(k+1)(k^2+2k+3k+6)}{3} \\ & =\frac{(k+1)(k+2)(k+3)}{3}. \\ & =r.h.s \end{aligned} \tag{2}$$

Thus by the principle of mathematical induction $\forall n \geq 1, S(n)$ is true. □

16. For every natural number n, it follows that $2^n + 1 \leq 3^n$.

Proof. Suppose $S(n) : 2^n + 1 \leq 3^n$, for $n \geq 1$. We need to show that $\forall n \geq 1, S(n)$ is true. **Basic step**: when $n = 1$, then $2^1 + 1 = 3 \leq 3^1$. Therefore, $S(1)$ is true.
Inductive step: suppose that when $S(k) : 2^k + 1 \leq 3^k$ is true. Thus, we want to prove that $S(k+1) : 2^{k+1} + 1 \leq 3^{k+1}$ is true. $l.h.s. = 2^{k+1} + 1 = 2 * 2^k + 1 = 2^k + 1 + 2^k \leq 3^k + 2^k < 3^k + 2 * 3^k = 3^{k+1} = r.h.s.$
Thus, $S(k+1) : 2^{k+1} + 1 \leq 3^{k+1}$ is true.
Thus by the principle of mathematical induction $\forall n \geq 1, S(n)$ is true. □

30. Here F_n is the nth Fibonacci number. Prove that

$$F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$$

Proof. We know that Fibonacci number is calculated by the following equation:

$$F_n = F_{n-1} + F_{n-2}, n \geq 3$$

with $F_1 = 1$, $F_2 = 1$.
Suppose $F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$, for $n \geq 1$. We need to show that $\forall n \geq 1, F_n$ is true.
Basic step: $F_1 = \frac{(\frac{1+\sqrt{5}}{2})^1 - (\frac{1-\sqrt{5}}{2})^1}{\sqrt{5}} = 1$, when $n = 1$.
$F_2 = \frac{(\frac{1+\sqrt{5}}{2})^2 - (\frac{1-\sqrt{5}}{2})^2}{\sqrt{5}} = 1$, when $n = 2$. Therefore, F_1, F_2 are true.
Inductive step: Suppose that $F_m = \frac{(\frac{1+\sqrt{5}}{2})^m - (\frac{1-\sqrt{5}}{2})^m}{\sqrt{5}}$ holds for any m , where $1 \leq m \leq k$, and we want to prove that $F_{k+1} = \frac{(\frac{1+\sqrt{5}}{2})^{k+1} - (\frac{1-\sqrt{5}}{2})^{k+1}}{\sqrt{5}}$ holds.

$$\begin{aligned}
l.h.s = F_{k+1} &= F_k + F_{k-1} \\
&= \frac{(\frac{1+\sqrt{5}}{2})^k - (\frac{1-\sqrt{5}}{2})^k}{\sqrt{5}} + \frac{(\frac{1+\sqrt{5}}{2})^{k-1} - (\frac{1-\sqrt{5}}{2})^{k-1}}{\sqrt{5}} \\
&= \frac{(\frac{1+\sqrt{5}}{2})^{k-1}(\frac{1+\sqrt{5}}{2}+1)}{\sqrt{5}} - \frac{(\frac{1-\sqrt{5}}{2})^{k-1}(\frac{1-\sqrt{5}}{2}+1)}{\sqrt{5}} \\
&= \frac{(\frac{1+\sqrt{5}}{2})^{k-1}(\frac{1+\sqrt{5}}{2})^2}{\sqrt{5}} - \frac{(\frac{1-\sqrt{5}}{2})^{k-1}(\frac{1-\sqrt{5}}{2})^2}{\sqrt{5}} \\
&= \frac{(\frac{1+\sqrt{5}}{2})^{k+1} - (\frac{1-\sqrt{5}}{2})^{k+1}}{\sqrt{5}} \\
&= r.h.s
\end{aligned} \tag{3}$$

Note that $\frac{1+\sqrt{5}}{2} + 1 = (\frac{1+\sqrt{5}}{2})^2$ and $\frac{1-\sqrt{5}}{2} + 1 = (\frac{1-\sqrt{5}}{2})^2$.
Thus, $F_{k+1} = \frac{(\frac{1+\sqrt{5}}{2})^{k+1} - (\frac{1-\sqrt{5}}{2})^{k+1}}{\sqrt{5}}$ holds.
Thus by the principle of mathematical induction $\forall n \geq 1, F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$ is true. □