Sol$^{\underline{n}}$ to HW2

1. Sample space size $= \binom{100}{3}$

Let a, b, c be 3 integers randomly picked.

Now $a+b+c$ is even if

a) all 3 are even.

b) two are ~~even~~ odd, the other one is ~~odd~~ even.

There are 50 even integers in $\{1, 2, \ldots, 100\}$

The rest are odd.

# of ways to choose 3 even integers $= \binom{50}{3}$

# of ways to choose 2 odd integers & one even integer $= \binom{50}{2} \times 50$

$\therefore$

$$\text{The probability} = \frac{\binom{50}{3} + 50 \times \binom{50}{2}}{\binom{100}{3}}.$$

2. We consider the following cases:

- $x \geq y$

$$f(x,y) = \frac{x+y}{2} + \frac{x-y}{2} = x$$
$$g(x,y) = x$$

- $x \leq y$

$$f(x,y) = \frac{x+y}{2} + \frac{y-x}{2} = y$$
$$g(x,y) = y$$

$$\Rightarrow f(x,y) = g(x,y) \; \forall \; (x,y) \in \mathbb{R} \times \mathbb{R}$$

3. Showing that f is one-to-one

Need to show that $f(x) = f(y) \Rightarrow x = y$.
(Contrapositive proof)

Let $f(x) = f(y)$

i.e. $\frac{4x+3}{x+2} = \frac{4y+3}{y+2}$

$$\Rightarrow x = y$$

Showing that f is onto.

Let $y \in \mathbb{R} - \{4\}$. Let x be the point in the domain s.t. $f(x) = y$

i.e. $\frac{4x+3}{x+2} = y$

i.e. $x = \frac{2y-3}{4-y}$ $\Rightarrow$ This is defined $\forall y \in \mathbb{R} - \{4\}$

$\therefore$ f is onto

Computing f^{-1}. [This exists since f is bijective)

Let $f^{-1}(x) = y$

i.e. $f(y) = x$.

i.e. $\frac{4y+3}{y+2} = x$

i.e. $y = \frac{3-2x}{x-4}$

4. $f(x) = ax+b$; $g(x) = cx^2+dx$.

$f(g(x)) = f(cx^2+dx) = a(cx^2+dx)+b$

$g(f(x)) = g(ax+b) = c(ax+b)^2+d(ax+b)$

$= a^2cx^2+(2abc+da)$
$+ b^2c+db.$

Now $f(g(x)) = g(f(x)]$ implies

i) $ac = a^2c$

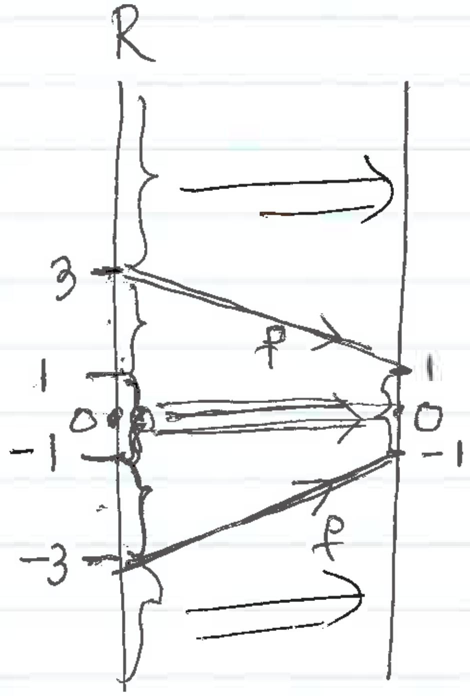
ii) $ad = 2abc+da$.

iii) $b = b^2c+db$.

Set $a=1$, $b=0$, c arbitrary & d arbitrary

This will realize $f(g(x)) = g(f(x))$.

5) Consider the following figure

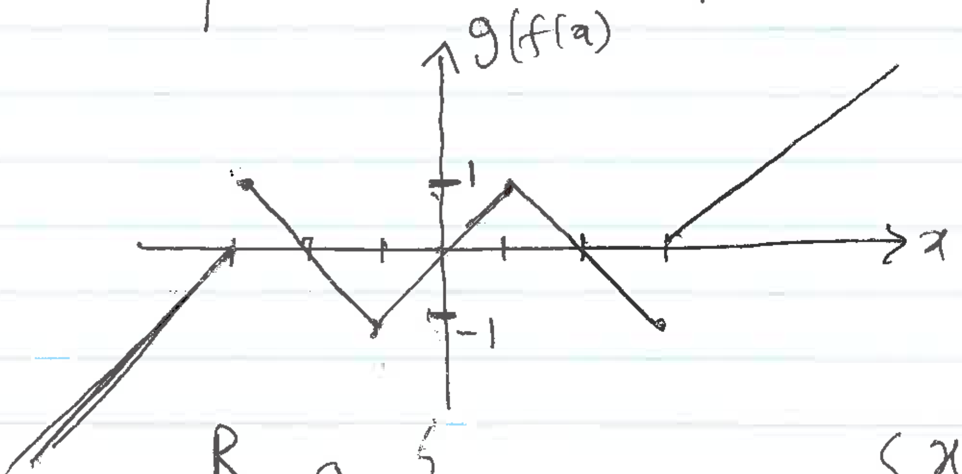


$f(3) = 1$

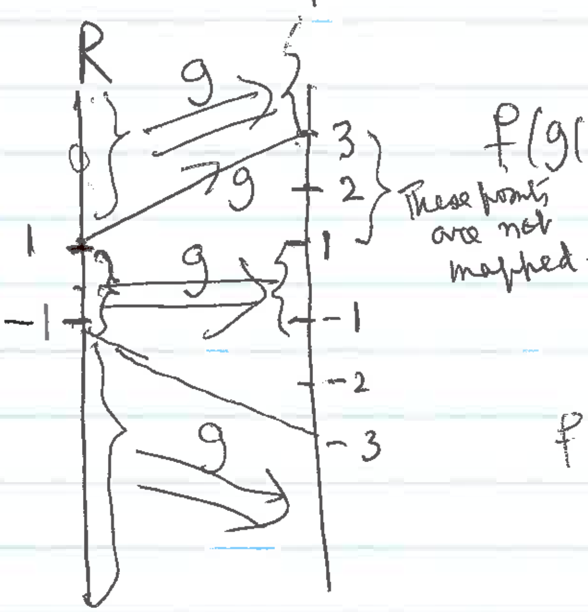
$f(-3) = -1$

Now
$$g(f(x)) = \begin{cases} x & x < -3 \\ -(x+2) & -3 \le x < -1 \\ x & -1 \le x \le 1 \\ -(x-2) & 1 < x \le 3 \\ x & 3 < x \end{cases}$$

The graph of this function is



Similarly



$$f(g(x)) = \begin{cases} x & x < -1 \\ x & -1 \le x \le 1 \\ x & 1 < x \end{cases}$$

$fg(x)$ is the identity function.

6. Easy: Straightforward application of the pigeonhole principle.

7. Here the # of pigeons = 5 & the holes are

(odd, odd); (odd, even); (even, odd), (even, even)

A point (4, 7) is mapped to hole (even, odd)
Similarly, (7, 4) is " " " (odd, even).

The pigeonhole principle implies that two of the five chosen points will have the same parity (odd or even) for both the x & the y-coordinate.

8. It follows from the fact that

$$\left(327^{400}\right)^{729} = \left(327^{729}\right)^{400}$$

$$\therefore \left(\left(327\right)^{400}\right)^{729} \bmod 919 = \left(327^{729}\right)^{400} \bmod 919.$$

To evaluate $a^n \bmod m$ for large n.

<u>One way:</u> <u>not very efficient.</u>

$$a^m \bmod n = (((a \bmod m) \times a \bmod m) \times a \bmod m)$$

....

- evaluate $(a \bmod m)$ then multiply with a & take mod m again.
- we get $a^2 \bmod m$.
- repeat the process.

The process will be repeated n ~~times~~ times.
For large n, this is prohibitive.

<u>Second Way</u> Let n in binary be

$$n = (b_1 b_2 \cdots b_k)_{\text{base } 2}, \quad b_i \text{ is either } 0 \text{ or } 1$$

$$\therefore a^n = a^{b_k} \cdot a^{2^1 b_{k-1}} \cdot a^{2^2 b_{k-2}} \cdots \cdot a^{2^{k-1} \cdot b_1}$$

$$\therefore a^n \bmod m = (a^{b_k} \bmod m) \cdot (a^{2^1 b_{k-1}} \bmod m) \cdots (a^{2^{k-1} \cdot b_1} \bmod m)$$

Now $a^{2^i} \bmod m = (a^{2^{i-1}} \bmod m)^2$

This is much more efficient.