Homework 1
MACM 101
March 20, 2015
Date due: March 27, 2015.

Use Induction principle to prove the following problems. The homework is due at the beginning of the class.

1. Prove the validity of the following Rule of Inference for all integers $n \geq 1$:

$$\begin{array}{ccc} p_1 & \rightarrow & p_2 \\ p_2 & \rightarrow & p_3 \\ \\ \ldots & \ldots & \ldots \\ \\ p_n & \rightarrow & p_{n+1} \\ & \neg p_{n+1} & \\ \hline & \neg p_1 & \end{array}$$

Proof: The above problem can be restated as:
$[(p_1 \Rightarrow p_2) \wedge (p_2 \Rightarrow p_3) \wedge (p_3 \Rightarrow p_4) \wedge \ldots \wedge (p_n \Rightarrow p_{n+1}) \wedge \neg p_{n+1}] \Rightarrow \neg p_1$.
Let $S(n)$ be the statement
$[(p_1 \Rightarrow p_2) \wedge (p_2 \Rightarrow p_3) \wedge (p_3 \Rightarrow p_4) \wedge \ldots \wedge (p_n \Rightarrow p_{n+1}) \wedge \neg p_{n+1}] \Rightarrow \neg p_1$.
Now
S(1):
$[(p_1 \Rightarrow p_2) \wedge \neg p_2] \Rightarrow \neg p_1$.
S(k):
$[(p_1 \Rightarrow p_2) \wedge (p_2 \Rightarrow p_3) \wedge \ldots \wedge (p_k \Rightarrow p_{k+1}) \wedge \neg p_{k+1}] \Rightarrow \neg p_1$.
S(k+1):
$[(p_1 \Rightarrow p_2) \wedge (p_2 \Rightarrow p_3) \wedge \ldots \wedge (p_k \Rightarrow p_{k+1}) \wedge (p_{k+1} \Rightarrow p_{k+2}) \wedge \neg p_{k+2}] \Rightarrow \neg p_1$.

Basic Case:
The contrapositive of $p_1 \Rightarrow p_2$ is $\neg p_2 \Rightarrow \neg p_1$. Together with $\neg p_2$ being true, according to Modus Ponens we have $\neg p_1$ is valid. Therefore S(1) is true.
Inductive Step:
Suppose $S(k) \Rightarrow \neg p_1$ is true. Now consider
S(k+1):
$[(p_1 \Rightarrow p_2) \wedge (p_2 \Rightarrow p_3) \wedge \ldots \wedge (p_k \Rightarrow p_{k+1}) \wedge (p_{k+1} \Rightarrow p_{k+2}) \wedge \neg p_{k+2}]$.
Applying the same strategy that we have used to prove the basic case, we can show that $(p_{k+1} \Rightarrow p_{k+2}) \wedge \neg p_{k+2} \Rightarrow \neg p_{k+1}$ is true. Together with the earlier

clauses, the left side of the implication for $S(k+1)$ is exactly the same as $S(k)$. Thus if $S(k) \Rightarrow \neg p_1$ is true, then $S(k+1) \Rightarrow \neg p_1$ is also true.

Thus we have showed that $S(1) \wedge (S(k) \Rightarrow S(k+1))$ is true for any $k \geq 1$. Therefore by the PMI, $\forall\ n \geq 1, S(n)$ is true.

2. Suppose n straight infinite lines lie on a plane in such a way that no two of the lines are parallel, and no three of the lines intersect at a single point. Show that this arrangement divides the plane into $\frac{n^2+n}{2} + 1$ regions.
   Proof:
   Basic Case For $n = 1$, a single straight infinite line on a plane can divide the plane into 2 regions which is equal to $\frac{1^2+1}{2} + 1$

   Inductive Step Suppose k straight infinite lines divide the plane into $\frac{k^2+k}{2}+1$ regions. Suppose now that we add one more line, say L, to k lines already there on the plane. We know that no two lines are parallel, and no three lines intersect at a single point (concurrent). This new line L, when added, will intersect each of the original k lines. These k intersection points lies on the boundaries of $k+1$ regions (old) determined by the k lines. The new line will split each of these old regions into two regions. The total number of new regions thus added after L is inserted is $k+1$. Thus we have
   $\frac{k^2+k}{2} + 1 + k + 1 = \frac{k^2+3k+2}{2} + 1 = \frac{(k+1)^2+(k+1)}{2} + 1$.

3. Prove that $S(n) : 3^0 + 3^1 + 3^2 + \ldots + 3^n = \frac{3^{n+1}-1}{2},\quad n \geq 0$ is true.
   Basic Case
   $S(0) : 3^0 = \frac{3^1-1}{2}$ is trivially true.
   Inductive Step
   Suppose $S(k) : 3^0 + 3^1 + 3^2 + \ldots + 3^k = \frac{3^{k+1}-1}{2}$ for any $k \geq 0$ is true.
   We now that $S(k+1)$ is also true.
   We can show that $3^0 + 3^1 + 3^2 + \ldots + 3^k + 3^{k+1} = \frac{3^{k+1}-1}{2} + 3^{k+1} = \frac{3*3^{k+1}-1}{2} = \frac{3^{k+2}-1}{2}$. This implies that $S(k+1)$ is also true if $S(k)$ is true.

   Thus we have showed that $S(0) \wedge (S(k) \Rightarrow S(k+1))$ is true for any $k \geq 0$. Therefore by the PMI, $\forall\ n \geq 0, S(n)$ is true.

4. Every integer $n \geq 14$ is expressible in the form $5a + 7b + 9c$ where a, b, c are nonnegative integers.

   Hints: a similar problem has been discussed in the class
   Basic Case
   $14 = 5 * 0 + 7 * 2 + 9 * 0$
   $15 = 5 * 3 + 7 * 2 + 9 * 0$
   $16 = 5 * 0 + 7 * 1 + 9 * 1$

$17 = 5 * 2 + 7 * 1 + 9 * 0$
$18 = 5 * 0 + 7 * 0 + 9 * 2$
Inductive Step Suppose we have a valid expression for the numbers from $14 + 5k$ to $14 + 5(k+1) - 1$ for $k \geq 0$.
Then for the next five numbers, that is the numbers from $14 + 5(k+1)$ to $14 + 5(k+2) - 1$ we could simple change the previous 5 expression by increasing the value by 5.

5. Define the following sequence of numbers: $a_1 = 2$ and for $n \geq 2, a_n = 5a_{n-1}$. Find a formula for a_n and then prove that its validity.

   Hints: We note that $a_2 = 2.5$, $a_3 = 2.5^2$, $a_4 = 2.5^3$, and so on.
   Looking at the values, we are guessing that $a_n = 2 * 5^{n-1}$. Now we need to show that our guess is a correct one. We will use the induction proof.
   Basic Case $S(1) : a_1 = 2 = 2 * 5^0$ is trivially true.
   Inductive Step Suppose $S(k) : a_k = 2 * 5^{k-1}$ is true. We need to show now that $S(k+1)$ is true.
   Now $S(k+1) : a_{k+1} = 5a_k = 5 * 2 * 5^{k-1} = 2 * 5^k$ is true.

   Thus we have showed that $S(1) \wedge (S(k) \Rightarrow S(k+1))$ is true for any $k \geq 1$. Therefore by the PMI, $\forall\ n \geq 1, S(n)$ is true.