Solution Set of Chapter 7, 8, 9

1 Chapter 7

4. Given an integer a, then a^2+4a+5 is odd if and only if a is even.
   Ans:
   We want to prove: if a is even, then a^2+4a+5 is odd.
   Suppose a is even, thus, $a=2k$, for some integer k.
   $a^2+4a+5=(2k)^2+4*2k+5=4k^2+8k+5=2(2k^2+4k+2)+1$, Thus, a^2+4a+5 is odd.

   We want to prove: if a^2+4a+5 is odd, then a is even.
   Suppose a^2+4a+5 is odd, since 5 is odd, thus, a^2+4a is even.
   Since $4a$ is even, thus, a^2 is even.
   Since a^2 is even, a is even.

   In summary, we get that given an integer a, then a^2+4a+5 is odd if and only if a is even.

6. Suppose $x,y\in\mathbb{R}$. Then $x^3+x^2y=y^2+xy$ if and only if $y=x^2$ or $y=-x$.
   Ans:
   We want to prove: if $x^3+x^2y=y^2+xy$, then $y=x^2$ or $y=-x$.
   Since $x^3+x^2y=y^2+xy$, we can get $x^3+x^2y-y^2-xy=(x^2-y)(x+y)=0$.
   Thus, we can get $y=x^2$ or $y=-x$.

   We also want to prove: if $y=x^2$ or $y=-x$, then $x^3+x^2y=y^2+xy$.
   Since $y=x^2$ or $y=-x$, thus $(y-x^2)(y+x)=0$.
   Since $(y-x^2)(y+x)=y^2+xy-x^2y-x^3=0$.
   Then we can get $y^2+xy=x^2y+x^3$.

   In summary, we get that suppose $x,y\in\mathbb{R}$. Then $x^3+x^2y=y^2+xy$ if and only if $y=x^2$ or $y=-x$.

10. If $a \in \mathbb{Z}$, then $a^3 \equiv a (mod 3)$.
    Ans:
    This is statement is true (corrected). We need to show that $a^3 - a$ is divisible by 3. We can write $a^3 - a = a(a^2 - 1) = a(a-1)(a+1)$. Any three consecutive integers will have one number divisible by 3. This proves the statement.

14. Suppose $a \in \mathbb{Z}$, Then $a^2 | a$ if and only if $a \in \{-1, 0, 1\}$.
    Ans:
    We want to prove: if $a^2 | a$, then $a \in \{-1, 0, 1\}$.
    Since $a^2 | a$, by definition, $a = k * a^2$, for some integer k and non-zero a.
    When $a \neq 0$, we have $1 = k * a$. Since $k, a \in \mathbb{Z}$, we have $a = 1, k = 1$ and $a = -1, k = -1$. Thus, we get $a \in \{-1, 0, 1\}$.

    We then want to prove: if $a \in \{-1, 0, 1\}$, then $a^2 | a$.
    Since $a \in \{-1, 0, 1\}$,
    when $a = -1$, $a^2 = 1$ divides $a = -1$, which is equal to -1.
    when $a = 1$, $a^2 = 1$ divides $a = 1$, which is equal to 1.
    when $a = 0$,

    0 can not divide 0!! Only if condition is not true.

20. There exists an $n \in \mathbb{N}$ for which $11 | (2^n - 1)$.
    Ans:
    Consider $n = 10$, then $2^n - 1 = 2^{10} - 1 = 1023 = 11 * 93$.
    Thus, the number $n = 10$ can make $11 | (2^n - 1)$.

22. If $n \in \mathbb{Z}$, then $4 | n^2$ or $4 | (n^2 - 1)$.
    If n is even, then $n = 2a$.
    $n^2 = (2a)^2 = 4a^2$, by definition, $4 | n^2$.

    If n is odd, then $n = 2a + 1$.
    $n^2 - 1 = (2a+1)^2 - 1 = 4a^2 + 4a = 4(a^2 + a)$, by definition, $4 | (n^2 - 1)$.

    In summary, we have shown that if $n \in \mathbb{Z}$, then $4 | n^2$ or $4 | (n^2 - 1)$.

28. Prove the division algorithm: If $a, b \in \mathbb{N}$, there exist unique integers q, r for which $a = bq + r$, and $0 \leq r < b$. (A proof of existence is given in Section 1.9, but uniqueness needs to be established too.)

Ans:
The proof of existence has been given in Section 1.9. We omit the detailed proof of existence and only provide the uniqueness.
Suppose we have two sets of integers q, r and $\bar{q}$, $\bar{r}$ that satisfies

$$a = bq + r \text{ and } 0 \leq r < b$$

and

$$a = b\bar{q} + \bar{r} \text{ and } 0 \leq \bar{r} < b.$$

We want to show that $\bar{q} = q$ and $\bar{r} = r$. We then have

$$b(\bar{q} - q) + \bar{r} - r = 0.$$

Since $0 \leq r < b$ and $0 \leq \bar{r} < b$, we have $-b < r - \bar{r} < b$. From this we obtain

$$-1 < \bar{q} - q = \frac{r - \bar{r}}{b} < 1.$$

Since $\bar{q} - q \in Z$, and $\bar{q} - q < 1$, this implies that

$$\bar{q} - q = 0.$$

This in turn implies that

$$\bar{r} - r = 0.$$

2 Chapter 8

Use the methods introduced in this chapter to prove the following statements.

2. Prove that $\{6n : n \in \mathbb{Z}\} = \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$
   Ans:
   Suppose $x \in \{6n : n \in \mathbb{Z}\}$, thus $x \in \mathbb{Z}$ and $x = 6n$.
   Thus $x = 2c$ and $x = 3k$ for some integer c, k.
   $x \in \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$
   We've shown $\{6n : n \in \mathbb{Z}\} \subseteq \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$
   Suppose $x \in \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$,
   By the definition of intersection, $x \in \mathbb{Z}$ and $x = 2k$ and $x = 3c$, for some integer k, c.
   Thus $x = 2k * 3c$,
   $x = 6b$, for some integer b. We get $x \in \{6n : n \in \mathbb{Z}\}$. We've shown $\{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\} \subseteq \{6n : n \in \mathbb{Z}\}$.
   In summary, we get $\{6n : n \in \mathbb{Z}\} = \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$.

6. Suppose A, B and C are sets. Prove that if $A \subseteq B$, then $A - C \subseteq B - C$.
   Ans:
   Suppose $x \in A - C$, by definition, we have $x \in A$ and $x \notin C$.
   Because $A \subseteq B$, thus $x \in B$.
   thus, $x \in B$ and $x \notin C$, i.e., $x \in B - C$.
   We have shown that $x \in A - C$ implies $x \in B - C$, so it follows that $A - C \subseteq B - C$.

20. Prove that $\{9^n : n \in \mathbb{Q}\} = \{3^n : n \in \mathbb{Q}\}$
    Ans:
    Suppose $x \in \{9^n : n \in \mathbb{Q}\}$, thus $x = 9^n$ and $n \in \mathbb{Q}$.
    $x = 3^{2n} = 3^k, k = 2n, k \in \mathbb{Q}$. It follows that $x \in \{3^n : n \in \mathbb{Q}\}$.
    We have shown that $x \in \{9^n : n \in \mathbb{Q}\}$ implies $x \in \{3^n : n \in \mathbb{Q}\}$, so it follows that $\{9^n : n \in \mathbb{Q}\} \subseteq \{3^n : n \in \mathbb{Q}\}$.

    Suppose $x \in \{3^n : n \in \mathbb{Q}\}$, thus, $x = 3^n$ and $n \in \mathbb{Q}$.
    $x = 3^n = 9^{n/2} = 9^k, k = n/2, k \in \mathbb{Q}$. It follows that $x \in \{9^n : n \in \mathbb{Q}\}$.
    We have shown that $x \in \{3^n : n \in \mathbb{Q}\}$ implies $x \in \{9^n : n \in \mathbb{Q}\}$, so it follows that $\{3^n : n \in \mathbb{Q}\} \subseteq \{9^n : n \in \mathbb{Q}\}$.

    In summary, we get $\{9^n : n \in \mathbb{Q}\} = \{3^n : n \in \mathbb{Q}\}$.

3 Chapter 9

Each of the following statements is either true or false. If a statement is true, prove it. If a statement is false, disprove it. These exercises are cumulative, covering all topics addressed in Chapters 1-9.

2. For every natural number n, the integer $2n^2 - 4n + 31$ is prime.
   Ans: False (corrected).
   $2n^2 - 4n + 31$ is not prime when $n = 31$. (Disproof through counterexample.)

18. If $a, b, c \in \mathbb{N}$, then at least one of $a - b, a + c$, and $b - c$ is even.
    Ans: True.
    Because at least two of a, b, c are both even or both odd, thus, at least one of $a - b, a + c$, and $b - c$ is even.

22. If p and q are prime numbers for which $p < q$, then $2p + q^2$ is odd.
Ans: True.
Because q is a prime number, and q is not the smallest prime, q is an odd number. Thus, q^2 is odd. $2p$ is even. Thus, $2p + q^2$ is odd.

34. If $X \subseteq A \cup B$, then $X \subseteq A$ or $X \subseteq B$.
Ans: True.
This is by definition.