Chapter 1: Sets

Definition of sets and their representations

A set is a collection of distinct objects. For example

- Set of students in a class
- Set of alphabets in English language

Each object of a set is called its element. A set can be represented in a variety of ways.

1. A set can be described by enclosing a list of distinct elements in braces. For example, $\{2,4,6,8\}$ is a set containing 4 elements, the numbers 2, 4, 6 and 8. Some sets have infinitely many elements. For example, consider the collection of all integers,

   $$\{....,-4,-3,-2,-1,0,1,2,3,....\}$$

   Here the dots indicate a pattern of numbers that continues forever in both the positive and the negative directions. A set is called an **infinite** set if it has infinitely many elements; otherwise it is called a **finite** set.

2. A special notation called **set-builder notation** is used to describe sets that are too complex to enumerate between the braces. Consider the infinite set of even integers $E = \{....,-6,-4,-2,0,2,4,6,...\}$. In the set-builder notation, this set can be written as

   $$E = \{n|n \text{ is an even integer }\} \text{ or } E = \{n : n \text{ is an even integer}\}$$

   Examples:

- $\{n | n \text{ is a prime integer}\} = \{2, 3, 5, 7, 11, 13, 17, ...\}$
- $\{x | x \text{ is an integer } |x| < 4\} = \{-3, -2, -1, 0, 1, 2, 3\}$

3. A set can be described by combining other sets using connectives defining particular set operations such as union ($\cup$), intersection ($\cap$), etc. Suppose $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{0, 2, 4, 6, 8\}$. Then the sets $C = A \cup B$ and $D = A \cap B$ are

$$C = \{0, 1, 2, 3, 4, 5, 6, 8\} \text{ and } D = \{2, 4, 6\}$$

4. A set can be defined by a recursive definition that identifies a few item explicitly, and then provides rules on how to obtain the other members from the ones already defined. Here is an example of defining the set B of all binary strings.

$$\text{`0'} \in B; \text{`1'} \in B$$
$$B = \{\text{`0'} \| u, \text{`1'} \| u : u \in B\}$$

Here $\|$ is used to denote concatenation of strings.

This approach will be explained in more detail later in the course.

We generally use capital letters to denote sets and lowercase letters to indicate the elements of a set.

Sets need not have just the numbers as elements. The set $B = \{T, F\}$ consisting of two letters, representing "true" and "false". The set $V = \{a, e, i, o, u\}$ consists of lowercase vowels. The set $D = \{(0,0), (1,0), (1,1), (0,1)\}$ contains the four corner points of a unit square in the $x - y$ coordinate plane. Thus $(0,0)$ is an element of D which is written as $(0,0) \in D$. Similarly, $(1,1) \in D$ etc., but $(1,2) \notin D$. Consider another set $A = \{1, \{1\}, \{2\}\}$. Here 1 and $\{1\}$ are two distinct elements of A. Therefore $1 \in A$, $\{1\} \in A$, but $2 \notin A$. Other examples of sets are $A = \{\{1,2\}, \{3,4,5,6\}, \{7\}\}$. $B = \{a, \{a\}, \{\{a\}\}\}$, $C = \{X \in A : |X| < 2\}$.

The symbol $\{x, y\}$ denotes the set containing two elements x and y. In this case, the set $\{x, y\}$ is sometimes called **unordered** set since $\{x, y\}$ and $\{y, x\}$ represent the same set. An **ordered** pair is a list (x, y) of two things x and y, enclosed in parentheses and separated by a comma. If x and y are distinct, $(x, y) \neq (y, x)$. We can generalize the above idea by writing $(x_1, x_2, ..., x_n)$ as an ordered list (called ordered n-tuple) consisting of n elements $x_1, x_2, \ldots, x_n$. Note that $\{(1,2),(2,1),(2,2)\}$ is the set containing three elements where each element is an ordered pair.

The **cardinality** or **size** of a set X is the number of elements X has. This number is denoted by $|X|$. If the set is infinite, the cardinality is infinite.

The **empty set** is the set that has no elements. We denote it as Φ. So $\Phi = \{\}$. Observe that $|\Phi| = 0$.

There are some sets or types of sets that come up so often that they are given special names and symbols.

- The natural numbers: $\mathbb{N} = \{1,2,3,4,5,6,...\}$
- The integers: $\mathbb{Z} = \{....,-3,-2,-1,0,1,2,3,4,...\}$
- The rational numbers: $\mathbb{Q} = \{x : x = \frac{m}{n}$, where $m,n \in \mathbb{Z}$ and $n \neq 0\}$
- The real numbers: $\mathbb{R}$ (set of all real numbers on the number line) (note: $\sqrt{2} \in \mathbb{R}$)

There are some other special sets that you will recall. Given two real numbers $x,y \in \mathbb{R}$ with $a \leq b$, we can form various intervals on the number line. A few of them are given below.

- Closed interval: $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
- Open interval : $(a,b) = \{x \in \mathbb{R} : a < x < b\}$ (the endpoints are not elements of the set)
- Half open interval: $[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$

Try the problems in section 1.1 of the text.

Set theory includes the following definitions:

Subsets: Set A is a subset of another set B if every element of A is also an element of B. We write $A \subseteq B$. We write $A \nsubseteq B$ if A is not a subset of B. In this case there exists at least one element of A which is not an element of B. Note that $\Phi \subseteq A$ and $A \subseteq A$ for any set A. A few more samples are given below.

- $\{2,3,7\} \subseteq \{2,3,,4,5,6,7\}$
- $\{2,3,7\} \nsubseteq \{2,4,5,6,7\}$
- $\{2,3,7\} \subseteq \{2,3,7\}$
- $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

Proper subset: A is a proper subset of B, denoted by $A \subset B$, if A is a subset of B and $|A| < |B|$. Thus $\Phi \subset A$ and $A \not\subset A$ for any non-empty set A.

Equal sets: Sets A and B are equal if $A \subseteq B$ and $B \subseteq A$.

Power set: If A is a set, the power set of A is another set, denoted as $\mathscr{P}(A)$, and defined to be the set of all subsets of A. In symbols, $\mathscr{P}(A) = \{X : X \subseteq A\}$. For example, suppose $A = \{1,2,3\}$. Then

$$\mathscr{P}(A) = \{\Phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Note that the cardinality (size) of the set $\mathscr{P}(A)$ is $2^{|A|}$ for any finite set A. We will see this in detail later. Observe one fact. We have seen that $\mathscr{P}(A) = \{\Phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ where $A = \{1,2,3\}$. Suppose $B = \{1,2,3,4\}$. Clearly, all the subsets of A are also the subsets of B. Thus we need to add, on top of $\mathscr{P}(A)$, all the subsets of B that contain the element 4. This is easy to achieve by listing the elements of $\mathscr{P}(B)$.

$$\begin{array}{ll}\mathscr{P}(B) = & \{\ \Phi,\ \{1\},\ \ \{2\},\ \ \{3\},\ \ \{1,2\},\ \ \{1,3\},\ \ \{2,3\},\ \ \{1,2,3\} \\ & \{4\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}\end{array}$$

Clearly, $|\mathscr{P}(B)| = 2|\mathscr{P}(A)|$.

Try the problems in sections 1.3 and 1.4 of the text.

Operations on sets

Let A and B be two arbitrary sets.

Union: The union of A and B is the set $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

Intersection: The intersection of A and B is the set $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Difference: The difference of A and B is the set $A - B = \{x : x \in A \text{ and } x \notin B\}$.

Symmetric Difference (page 136 of Grimaldi's book) The symmetric difference of A and B is the set $A \bigtriangleup B = \{x : (x \in A \text{ and } x \notin B) \text{ or } (x \notin A \text{ and } x \in B)\}$. It can be shown that $A \bigtriangleup B = (A - B) \cup (B - A)$.

Example 1.5 of the text:

Suppose $A = \{a, b, c, d, e\}$, $B = \{d, e, f\}$ and $C = \{1, 2, 3\}$.

1. $A \cup B = \{a, b, c, d, e, f\}$
2. $A \cap B = \{d, e\}$
3. $A - B = \{a, b, c\}$
4. $B - A = \{f\}$
5. $A \cap C = \Phi$
6. $A - C = A$
7. $(A \cap C) \cup (A - C) = \{a, b, c, d, e\}$

Example 1.6 of the text:

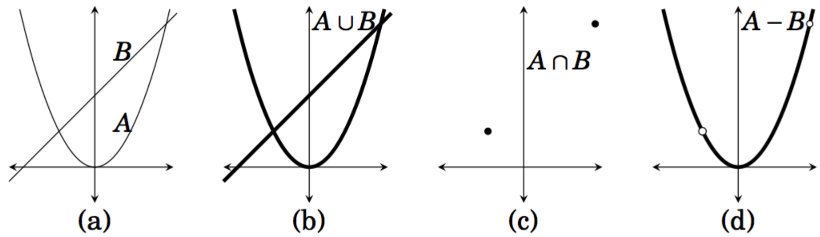


Figure 1.5. The union, intersection and difference of sets A and B

Complement: The definition requires the concept of **universal set**. The universal set is the set of all elements from which the members of each set must be chosen. Consider the set of prime numbers $P = \{2,3,5,7,11,13,...\}$. We can easily claim that $422 \notin P$. We have an unstated assumption that $P \subseteq \mathbb{N}$ because $\mathbb{N}$ is the most natural setting in which to discuss the prime numbers. In this context $\mathbb{N}$ is the universal set.

The complement of a set A with respect to the universal set U is the set $\overline{A} = U - A$.

The complement of P defined above is $\overline{P} = \{1,4,6,8,9,10,....\}$. Thus $\overline{P}$ is the set of composite numbers (which can be factored) and 1.

Example 1.8 Let $A = \{(x,x^2) : x \in \mathbb{R}\}$ be the graph of the equation $y = x^2$. Figure 1.6(a) shows A in its universal set $\mathbb{R}^2$. The complement of A is $\overline{A} = \mathbb{R}^2 - A = \{(x,y) \in \mathbb{R}^2 : y \neq x^2\}$, illustrated by the shaded area in Figure 1.6(b).

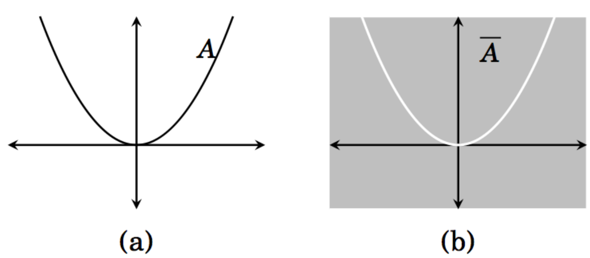


Figure 1.6. A set and its complement

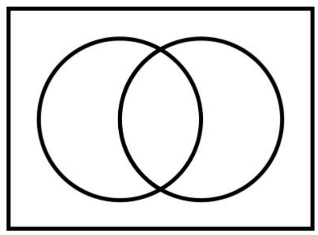
Some identities: Let A and B be two arbitrary sets. We can the show that

1. $A \cup B = B \cup A$
2. $A \cap B = B \cap A$
3. $A - B \neq B - A$
4. if $A \cap B = \phi$, then A and B have no elements in common.
   Thus $|A \cup B| = |A| + |B|$.
5. If $B \subseteq A$ then
   (a) $A \cup B = A$
   (b) $A \cap B = B$
   (c) $\overline{A} \subseteq \overline{B}$

Try the problems in sections 1.5 and 1.6 of the text.

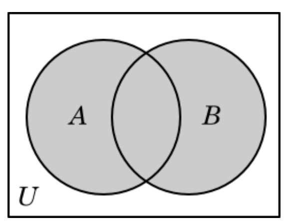
Venn Diagram

This concept was introduced by John Venn. It is the diagramatic representation of sets. The universal set is represented by a rectangle and the sets are represented by circles inside the rectangles.

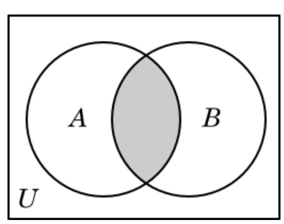


In the above Diagrams the circles represent two sets having some elements in common **Set operations in Venn Diagram:** Let A and B be two arbitrary sets. Let U be the universal set.

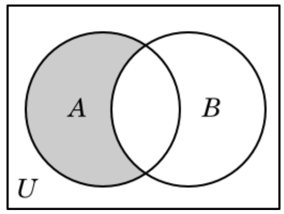
- $A \cup B$: The shaded part represents the union.



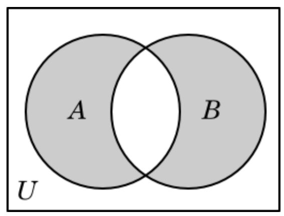
- $A \cap B$: The shaded part represents the intersection.



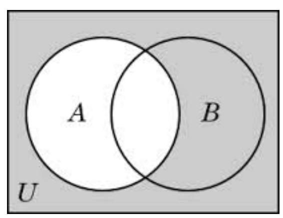
- $A - B$: the shaded part represents the difference.



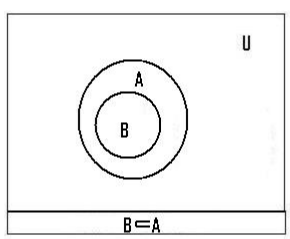
- $A \triangle B$: the shaded part represents the symmetric difference.



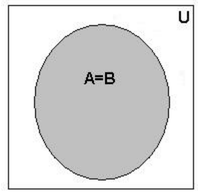
- $\overline{A} = U - A$: The shaded part represents the complement.



- $B \subseteq A$



- $A = B$



Try the problems in section 1.7 of the text.

Indexed Sets

Suppose we are interested in working with three sets. Instead of using A, B and C as the names of the set, we can use indexed sets A_1, A_2 and A_3. This way we can label many sets in consistent manner.

Generalized Union and Intersection: Suppose $A_1, A_2, ..., A_n$ are sets. Then

$$\begin{aligned} A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = & \quad \{x : x \in A_i \text{ for at least one set } A_i, \text{ for } 1 \leq i \leq n\}, \\ A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n = & \quad \{x : x \in A_i \text{ for every set } A_i, \text{ for } 1 \leq i \leq n\}, \end{aligned}$$

We can write

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n; \quad \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n$$

Other ways to write the above sets:

$$\bigcup_{i \in \{1,2,...,n\}} A_i = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n; \quad \bigcap_{i \in \{1,2,...,n\}} A_i = A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n$$

$$\bigcup_{1 \leq i \leq n} A_i = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n; \quad \bigcap_{1 \leq i \leq n} A_i = A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n$$

$$\bigcup_{i \in I} A_i = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n; \quad \bigcap_{i \in I} A_i = A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n; I = \{1, 2, 3, ..., n\}$$

These definitions can be applied to infinite collections of sets as well. For instance suppose that $B_n = \{kn : k = 2, 3, 4, ...\}$ (set of multiples of n greater than n). Then

$$\bigcup_{n=2}^{\infty} B_n = B_2 \cup B_3 \cup B_4 \cup .... = \{4, 6, 8, 9, 10, 12, 14, 15, ...\}$$

which is the set of composite positive numbers (numbers which can be factored).

Try the problems in section 1.8 of the text.

Manipulations of sets

Notice: This topic is not covered in the text. The topic is covered in Grimaldi's book (Section 3.2) and in Rosen's book (section 2.2). One way of manipulating sets is to express them in different ways. The purpose may be to simplify the representation, or to compare two different representations. Laws of set theory are established to facilitate the manipulations of sets. These laws allow transformation of compound expressions of sets into different but equivalent expressions.

Laws Of Set Theory

1. Double Complement

   $\overline{\overline{A}} = A$

2. De-Morgan's Law

   (a) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

   (b) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

3. Commutative Law

   (a) $A \cup B = B \cup A$

   (b) $A \cap B = B \cap A$

4. Associative Law

   (a) $A \cup (B \cup C) = (A \cup B) \cup C$

   (b) $A \cap (B \cap C) = (A \cap B) \cap C$

5. Distributive Law

   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

6. Idempotent Law

   (a) $A \cup A = A$

   (b) $A \cap A = A$

7. Identity Law

(a) $A \cup \phi = A$

(b) $A \cap \phi = \phi$

8. Inverse Law

(a) $A \cup \overline{A} = U$

(b) $A \cap \overline{A} = \phi$

9. Dominating Law

(a) $A \cup U = U$

(b) $A \cap \phi = \phi$

10. Absorbtion Law

(a) $A \cup (A \cap B) = A$

(b) $A \cap (A \cup B) = A$

A Venn diagram is simply a graphical visualization of membership association. This allows us to understand certain situations. Clearly, Venn diagram looses its effectiveness when the number of involved sets is more than 3. All the above mentioned laws can be established using element arguments. Below formal proofs of the second of DeMorgan's Laws and the first Distributive Law are provided.

Showing $\overline{A \cap B} = \overline{A} \cup \overline{B}$ (DeMorgan's Law)

The argument consists of two parts. We first show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$, and then show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. The result then follows from the definition of set equality.

Part 1: Let x be an arbitrary element of $\overline{A \cap B}$. This means that $x \notin A \cap B$, which implies that $x \notin A$ and $x \notin B$. Therefore, $x \in \overline{A}$ or $x \in \overline{B}$, that is $x \in \overline{A} \cup \overline{B}$. Thus we see that every element of $\overline{A \cap B}$ is an element of $\overline{A} \cup \overline{B}$. The definition of subset implies that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

Part 2: Let an arbitrary element $x \in (\overline{A} \cup \overline{B})$. This means that $x \in \overline{A}$ or $x \in \overline{B}$, which implies that $x \notin A$ and $x \notin B$. This implies that $x \notin A \cap B$. Therefore, $x \in \overline{A \cap B}$. The definition of subset implies that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. Notice that the arguments for the second part are the same as the first part, except in reverse order.

Showing $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ **(Distributive Law)**

Again the arguments consist of two parts. We first show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$, and then show that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Part 1: Let x be an arbitrary element of $A \cup (B \cap C)$. This means that $x \in A$ or $x \in B \cup C$. If $x \in A$, $x \in (A \cup B)$ and $x \in (A \cup C)$. This implies that $x \in (A \cup B) \cap (A \cup C)$. Therefore, when $x \in A$, $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

If $x \in B \cap C$, $x \in B$ and $x \in C$. This implies that $x \in (A \cup B) \cap (A \cup C)$. Therefore, when $x \notin A$, then also $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Part 2: Let x be an arbitrary element of $(A \cup B) \cap (A \cup C)$. Therefore, $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$, $x \in A \cup (B \cap C)$. If $x \notin A$, $x \in B$ and $x \in C$. This means that $x \in B \cap C$, that is, $x \in A \cup (B \cap C)$. Therefore, $(A \cup B) \cap (A \cup C) \subseteq A \cap (B \cup C)$.

The element arguments given here have four components. First choose the element, then apply the definitions which is then followed by drawing inferences. The last component is about reaching the conclusion.

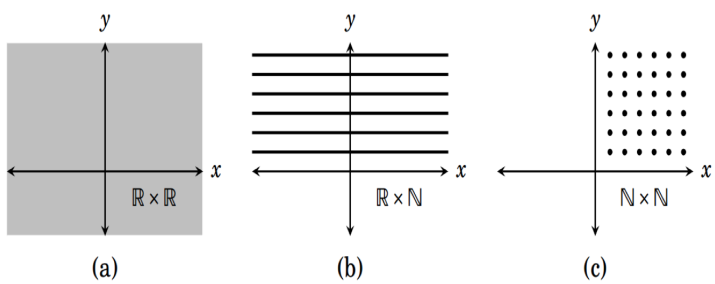
The Cartesian Product

The **Cartesian product** of two sets A and B is another set, denoted as $A \times B$ and defined as $A \times B = \{(a,b) : a \in A, b \in B\}$.

Note that generally $A \times B$ is not the same as $B \times A$.

The name "Cartesian product" comes from a geometric interpretation. If for instance $A = B = \mathbb{R}$, $A \times B$ can be interpreted as all the points in the plane (Fig. (a) below), since a point in the plane is uniquely described by an ordered pair of real numbers, namely it Cartesian coordinates - x-coordinate and y-coordinate. Figure (b) is $A \times B$ where $A = \mathbb{R}$ and $B = \mathbb{N}$. Figure (c) is $\mathbb{N} \times \mathbb{N}$.

The Cartesian product of a set with itself, i.e. $A \times A$, may also be denoted by A^2.



Try the problems in section 1.2 of the text.

Sections 1.9 and 1.10 of the text will be skipped for the time being. We will visit them later.
