MACM 101 - D2 : Quiz 5 (Full Marks 22) Time: 30 minutes
March 27, 2015

The solution set for D1 section is very similar to that of D2 section.

1. (6 points) Let R be a relation on $\mathbb{N}$, the set of positive integers. Determine the entries of the following table on the properties of $R \in \{<, |\}$.

Relation on $\mathbb{N}$	$<$	$\vert$
reflexive	F	T
symmetric	F	F
antisymmetric	T	T
transitive	T	T

2. (6 points) Let R be the relation on $A = \{1,2,3,4,5\}$ where $R = \{(1,1),(1,3),(1,4),(2,2),(1,5),(3,1),(3,3),(3,4),(4,1),(4,3),(4,4),(5,5),(5,1)\}$.

   (a) Write the matrix for R. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1:1 & 0 & 1 & 1 & 1 \\ 2:0 & 1 & 0 & 0 & 0 \\ 3:1 & 0 & 1 & 1 & 0 \\ 4:1 & 0 & 1 & 1 & 0 \\ 5:1 & 0 & 0 & 0 & 1 \end{pmatrix}$

   (b) Draw the directed graph for R.
   See Fig.1.

   (c) Find the equivalence classes for the partition of A given by R.

   R is not an equivalence relation on A. For example the transitive arc $(3,5)$is missing. So there is no equivalence class for the partition of A given by R.

   (**Note D1 group students**: the given relation in D1-exam is an equivalence relation.)

3. Answer any one of the following two.

(a) (6 points) Let R be the binary relation on $\mathbb{N}$ defined as

$$R = \{(a,b) \in \mathbb{N} \times \mathbb{N} | a \equiv b (mod\ 4\ )\}.$$

Show that R is an equivalence relation on $\mathbb{N}$ and find the different equivalence classes.

R is reflexive: For each element $a \in \mathbb{N}$, $(a,a) \in R$, since (a-a) is divisible by 4.

R is symmetric: if $(a,b) \in R$, this means $a \equiv b(mod\ 4\ )$. So we can get $b \equiv a(mod\ 4\ )$. Thus, $(b,a) \in R$.

R is transitive: if $(a,b) \in R$ and $(b,c) \in R$, this means $a \equiv b(mod\ 4\ )$ and $c \equiv c(mod\ 4\ )$. So we can get $a \equiv c(mod\ 4\ )$. Thus, $(a,c) \in R$.

Different equivalence classes: $\{4k, k \in \mathbb{N}\}$,$\{4k+1, k \in \mathbb{N}\}$,$\{4k+2, k \in \mathbb{N}\}$ and $\{4k+3, k \in \mathbb{N}\}$.

(b) (6 points) Consider the relation "*divides*" on the set $\{2,3,4,5,6,12,20\}$.

i. Show that "*divides*" is a partial order.

The relation "divide" is $\{(2,2),(2,4),(2,6),(2,12),(2,20),(3,3),(3,6),(3,12),$ $(4,4),(4,12),(4,20),(5,5),(5,20),(6,6),(6,12),(12,12),(20,20)\}$
We can see the relation satisfies reflexive, antisymmetric and transitive property. So the relation "divide" is a partial order.

ii. Draw the Hasse diagram.

See Fig.2.

iii. Determine the minimal, maximal elements of the set.
Minimal element is $\{2,3,5\}$ and maximal element is $\{12,20\}$.

4. (4 points) Find the smallest equivalence relation on $\{1,2,3,4\}$ that contains $(1,2)$ and $(1,3)$.

The relation $\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(4,4)\}$ is an equivalence relation of smallest cardinality that contains the edges

$(1,2)$ and $(1,3)$. Any smaller size relation will not be an equivalence relation.

5. (Bonus Question) (5 points) Suppose R is an equivalence relation on a set A, with four equivalence classes. How many different equivalence relations S on A are there for which $R \subseteq S$?

   Let $\{A\},\{B\},\{C\},\{D\}$ be the four equivalence classes realized by R. Any equivalence relation $S, R \subset S$, will result in a merging of some of these sets. Therefore, the number of such equivalence relations is equal to the number of ways merging these four sets. This is the same as the number of partitions of a set with four elements.

   The possible partitions of $\{A,B,C,D\}$ are:

   Partition of size 1: $\{A,B,C,D\}$

   Partition of size 2: $[\{A,B,C\},\{D\}],[\{A,B,D\},\{C\}],[\{A,C,D\},\{B\}],$ $[\{B,C,D\},\{A\}]$ $[\{A,B\},\{C,D\}],[\{A,C\},\{B,D\}],[\{A,D\},\{B,C\}]$

   Partition of size 3: $[\{A,B\},\{C\},\{D\}],[\{A,C\},\{B\},\{D\}],[\{A,D\},\{B\},\{C\}],$ $[\{B,C\},\{A\},\{D\}],[\{B,D\},\{A\},\{C\}],[\{C,D\},\{A\},\{B\}]$

   Partition of size 4: $[\{A\},\{B\},\{C\},\{D\}]$

   Thus, there are 15 such equivalence relations.

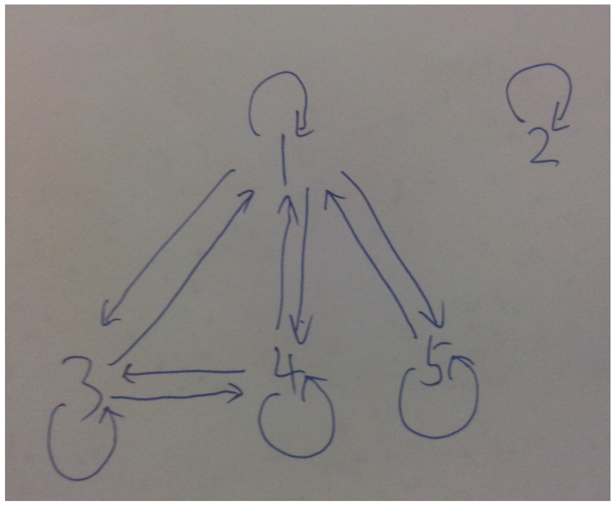

Figure 1: Directed graph for *R*

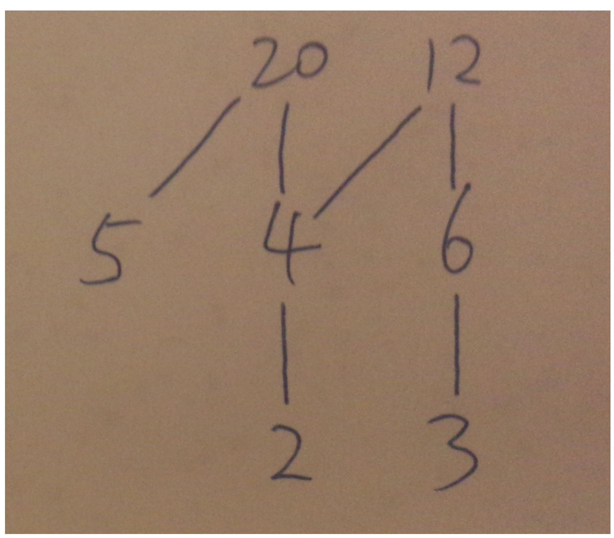

Figure 2: Hasse diagram