MACM 101 : Quiz 3 (Full Marks 45) Time: 40 minutes

1. (10)This problem concerns list made from the letters A,B,C,D,E,F,G,H,I,J.

   (a) How many length-5 lists can be made from these letters if repetition is not allowed and and the list must begin with a vowel.
   $3*9*8*7*6$ {There are three vowels (A, E and I) available for position 1}

   (b) How many length-5 lists can be made from these letters if repetition is allowed and the list must begin and end with a vowel?
   $(3*10*10*10*3)$ { There are 3^2 ways to fill the first and the last positions. }

   (c) How many length-5 lists can be made from these letters if repetition is not allowed and the list must contain exactly one A?
   $5*9*8*7*6$ { A can be placed in 5 different places. }

2. (10) In how many ways a gambler draw six cards from a standard deck and get

   (a) a flush (six cards of the same suit)?
   $4*C(13,6)$

   (b) four aces?
   $C(4,4)*C(48,2)$

   (c) four of a kind?
   $13*C(4,4)*C(48,2)$

   (d) three aces and two jacks?
   $C(4,3)*C(4,2)*C(44,1)$

   (e) three aces and a pair?
   $C(4,3)*(13-1)*C(4,2)*C(44,1)$

   (f) three of a kind and a pair?
   $C(13,1)*C(4,3)*C(12,1)*C(4,2)*C(44,1)$

   (g) three pairs?
   $C(13,3)*C(4,2)*C(4,2)*C(4,2)$

3. (10) Consider making lists from the symbols A,B,C,D,E,F,G. How many length-4 lists are possible if:

   (a) repetition is allowed?
   $7*7*7*7$

   (b) repetition is not allowed and the list must contain E?
   $4*6*5*4$

   (c) repetition is allowed and the list must contain an E?
   $7*7*7*7-6*6*6*6$

4. (10) Consider the lists of length 6 made with the symbols P, R, O, F, S, where repetition is allowed. How many such lists can be made if the list must end in an S and the symbol is used more than once.
$5^5 - 5*4^4 - 4^5$

{ **This problem was discussed in the class.** }

5. (10) The problem concerns lists made from the symbols A,B,C,D,E,F,G,H,I.

(a) How many length-5 lists can be made if repetition is not allowed and the list is in alphabetical order? (Example: BDEFI or ABCGH, but not BACGH)
C(9,5) { **Every subset of size 5 realizes one list in alphabetical order.** }

(b) How many length-5 lists can be made if repetition is not allowed and the list is not alphabetical order.
$9*8*7*6*5 - C(9,5)$

6. (10) Compute how many 9-digit numbers can be made from the digits $1,2,3,4,5,6,7,8,9$ if the repetitions are not allowed, and all the odd digits occur first (on the left) followed by the even digits (i.e. as in 137598264, but not 123456789).
$5!*4!$ { **There are 5 odd digits and 4 even digits.** }

7. (5) Using only pencil and paper, find the value of

   (a) $C(10,8)$
   equal to $C(10,2) = 10*9/2$

   (b) $C(20,18)$
   equal to $C(20,2) = 20*19/2$

   (c) $\frac{15!}{3!12!}$
   $15*14*13/3*2*1 = 5*7*13$

   (d) $\frac{9!}{2!6!}$
   $9*8*7/2$

   (e) $\frac{10!}{8!}$
   $10*9$

   (f) $\frac{20!}{17!}$
   $20*19*18$

8. (5) Define binomial theorem. Use the binomial theorem to find the coefficient of

   (a) x^6y^8 in $(x+y)^{14}$
   $C(14,6)$

   (b) x^7 in $(x-2)^{13}$
   $C(13,7)\times(-2)^6$

   (c) x^6y^3 in $(3x-2y)^9$.
   $C(9,6)*3^6*(-2)^3$

   (d) x^9y^3 in $(4x-2y)^{12}$.
   $C(12,9)*4^9*(-2)^3$

(e) x^6 in $(2+x^2)^9$.
$C(9,3) \times 2^9$

(f) x^8y^5 in $(x+y)^{13}$.
$C(13,8)$

(g) x^8 in $(x+2)^{13}$.
$C(13,8) * 2^5$

9. (5) Let A, B, be subsets of a finite set U. Then show that (use Venn diagram to formulate your arguments)

   (a) $|A \cup B| = |A| + |B| - |A \cap B|$,

   (b) $|A \cap B| \leq min\{|A|, |B|\}$

   (c) $|A - B| = |A| - |A \cap B|$,

10. (5) Use counting argument to show that for any integer k, $1 \leq k \leq n$, $C(n+1,k) = C(n,k-1) + C(n,k)$.

    Let a be an element of a set containing $n+1$ elements. In order to select k elements out of $n+1$ elements, we have two situations. First if this "a" element is selected as one of the k elements, then we select the other $k-1$ elements out of n remaining elements. We can do this in $C(n,k-1)$ different ways. The other situation is that this "a" element is not selected as one of the k elements, then we need to select all the k elements from the remaining n elements. This can be done in $C(n,k)$ ways. Thus, $C(n,k-1) + C(n,k)$ is another way to calculate $C(n+1,k)$.

11. (5) Suppose a set B has the property that $|\{X : X \in \mathscr{P}(B), |X| = 5\}| = 21$. What is $|B|$?

    Suppose $|B| = x$. Then we have $C(x,5) = 21$. Thus we have $\frac{x*(x-1)*(x-2)*(x-3)*(x-4)}{5*4*3*2*1} = 21$, thus we want to make $5*4*3*2*1*21$ to be the multiplication of 5 continuous numbers.$5*4*3*2*1*21 == 5*4*3*2*1*3*7 = 7*6*5*4*3$. Thus $|B| = 7$.

12. (5) Suppose a set B has the property that $|\{X : X \in \mathscr{P}(B), |X| = 6\}| = 28$. Find $|B|$.

    Similar as the previous problem. Suppose $|B| = x$. Then we have $C(x,6) = 28$. $x = 8$

13. (5) What is the value$|\{X : X \in \mathscr{P}(\{0,1,2,3,4,5,6,7,8,9\}), |X| < 4\}| =?$.

    $C(10,0) + C(10,1) + C(10,2) + C(10,3)$

    { Note that
    $|\{X : X \in \mathscr{P}(\{0,1,2,3,4,5,6,7,8,9\}), |X| < 4\}| =$

$|\{X : X \in \mathscr{P}(\{0,1,2,3,4,5,6,7,8,9\}), |X| = 0\}| +$
$|\{X : X \in \mathscr{P}(\{0,1,2,3,4,5,6,7,8,9\}), |X| = 1\}| +$
$|\{X : X \in \mathscr{P}(\{0,1,2,3,4,5,6,7,8,9\}), |X| = 2\}| +$
$|\{X : X \in \mathscr{P}(\{0,1,2,3,4,5,6,7,8,9\}), |X| = 3\}|.$
}

14. (5) Is the following statement true or false? Explain. If $A_1 \cap A_2 \cap A_3 = \emptyset$, then $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$.

    The statement is false. A counter is the following. $A = \{1,2\}$,$B = \{2,3\}$,$C = \{0\}$. $A_1 \cap A_2 \cap A_3 = \emptyset$.
    $|A_1 \cup A_2 \cup A_3| == |\{0,1,2,3\}| = 4$.
    $|A_1| + |A_2| + |A_3| = 2+2+1 = 5$

15. (5) Determine how many zeros are there at the end when 77! is evaluated.

    We would have a zero at the end if we have $2*5$ in the factorial. Since we have way too many 2 in 77!. So all we have to do is to count the number of 5, for example for $5, 10, 15, 20...$ we have one 5, for $25, 50, 75$, we have two 5 for each of them. Add them up and we will have 18 zeros.

16. (5) Determine how many zeros are there at the end when 150! is evaluated.

    The analysis is similar to the previous problem. The answer should be $6*6+1 = 37$

17. (5) Use tree diagram to generate length-3 lists using the elements of $A = \{0,1\}$.

    Similar tree diagram is given in the text.

18. (5) Formally describe the multiplication rule.

    Look at the text for the definition.

19. (10) This problem involves lists made from the letters T,H,E,O,R,Y, with repetition allowed.

    (a) How many 4-letter lists are there that don't begin with T, or don't end in Y?
    Let set A_T contains lists that begin with T. Let set A_Y contains lists that end with Y. Let A_{TY} contains lists that begin with T and end with Y. There are 6^4 possible ways to select length-4 lists. Therefore, the answer is $6^4 - |A_T| - |A_Y| + |A_{TY}|$. We applied the principle inclusion and exclusion.

    (b) How many 4-letter lists are there in which the sequence of letters T,H,E appears consecutively?
    In this case T of THE can occupy positions one or two only. The remaining position can be filled in 6 ways. Thus the answer is $2*6$.

(c) How many 5-letter lists are there in which the sequence of letters T,H,E appears consecutively?

In this case the answer is $3 * 6^2$

20. (10) How many 7-digit binary strings begin in 1 or end in 1 or have exactly four 1s?

Let A_1 be the set of all strings that begin with 1. Let A_2 be the set of all strings that end in 1. Let A_{12} be the set of strains that begin and end with 1. Let A_3 : Set of all strings that have exactly four 1s, and the first and the last positions are 0.

$|A_1| = |A_2| = 2^6$, and $|A_3| = C(5,4)$.

The answer is $|A_1| + |A_2| - |A_{12}| + |A_3|$.

21. (10) A 3-card hand is dealt off of a standard 52-card deck. How many different such hands are there for which all 3-cards are red or all three cards are face cards (i.e. J, Q or K of any suit)?

$C(26,3) + C(12,3) - C(6,3)$

22. (10) A 4-letter list is made from the letters L,I,S,T,E,D according to the following rule: Repetition is allowed, and the first two letters on the list are vowels or the list ends in D. How many such lists are possible?
$2^2 * 6 * 6 + 6 * 6 * 6 - C(2 * 2 * 6)$

23. (10) How many 7-digit binary strings (0010100, 1101011, etc.) have an odd number of 1s?
C(7,1)+C(7,3)+C(7,5)+C(7,7)