MACM 101 Midterm Exam
February 27, 2015.

Name:	
Student Number:	
Signature:	

Q. 1(a)	/ 5
Q. 1(b)	/ 5
Q. 2(a)	/5
Q. 2(b)	/ 5
Q. 2(c)	/ 5
Q. 3	/20
Q. 4	/5
Q. 5 (Bonus)	/10
TOTAL	/50

1. Questions on Set Theory.

   a) (5 points) Let A, B, C be three arbitrary sets of the universal set U. Proof the following using the element containment argument.

      i. $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$.

         - **<u>Ans:</u>** Suppose x is an element of $\overline{A \cap B \cap C}$. In this case x is not an element of $A \cap B \cap C$. Therefore, x does not belong to either A, B, or C. Suppose x is not an element of A. This implies that x is an element of $\overline{A}$, and therefore, is an element of $\overline{A} \cup \overline{B} \cup \overline{C}$. Thus $\overline{A \cap B \cap C} \subseteq \overline{A} \cup \overline{B} \cup \overline{C}$.

           We want to show that $\overline{A \cap B \cap C} \supseteq \overline{A} \cup \overline{B} \cup \overline{C}$. Let x be an element of $\overline{A} \cup \overline{B} \cup \overline{C}$. Without any loss of generality we assume that $x \in \overline{A}$. This means that $x \notin A$, and therefore, $x \notin A \cap B \cap C$. This says that $x \in \overline{A \cap B \cap C}$. Therefore, $\overline{A \cap B \cap C} \supseteq \overline{A} \cup \overline{B} \cup \overline{C}$.

           We have thus proved the statement $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$.

      ii. $\overline{A \cup B \cup C} = \overline{A} \cap \overline{B} \cap \overline{C}$.

         - **<u>Ans:</u>** Suppose x is an element of $\overline{A \cup B \cup C}$. In this case x is not an element of $A \cup B \cup C$. Therefore, x does not belong to any of the sets A, B, and C. This implies that x is an element of $\overline{A} \cap \overline{B} \cap \overline{C}$. Therefore, x is an element of $\overline{A} \cap \overline{B} \cap \overline{C}$. Thus , $\overline{A \cup B \cup C} \subseteq \overline{A} \cap \overline{B} \cap \overline{C}$.

           We want to show that $\overline{A \cup B \cup C} \supseteq \overline{A} \cap \overline{B} \cap \overline{C}$. Let x be an element of $\overline{A} \cap \overline{B} \cap \overline{C}$. Therefore, $x \notin A \cup B \cup C$, i.e. $x \in \overline{A \cup B \cup C}$. Thus, $\overline{A \cup B \cup C} \supseteq \overline{A} \cap \overline{B} \cap \overline{C}$.

           We have thus proved the statement $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$.

   b) (5 points) Which of the following are always true (A, B are arbitrary sets)? Yes-No answers.

      i. if $A \subseteq B$ and $B \subseteq A$, then $A = B$. **YES**

      ii. $\emptyset \in \{\emptyset\}$. **YES**

      iii. $\emptyset \in A$. **NO** Consider $A = \{2, 3\}$

      iv. $\emptyset \subseteq A$. **YES**

      v. $A - B = \overline{(\overline{A} - \overline{B})}$. **NO** Consider $A = B$.

      vi. $(A - B) \cap (B - A) = \emptyset$. **YES**

      vii. $\emptyset \times A = \emptyset$. **YES**

      viii. $A \times B = B \times A$. **NO**

      ix. $\overline{A \cup B} = \overline{A} \cup \overline{B}$. **NO**

2. Questions on Logic.

a) (5 points) Give reasons for each step in the proof of the following. A proof based on the truth truth table is not acceptable.

   i. $[(p \vee q) \wedge (p \vee \neg q)] \vee q \equiv p \vee q$.

      - **Ans:** The argument goes as follows:
      $$\begin{array}{rcll} [(p \vee q) \wedge (p \vee \neg q)] \vee q & \equiv & [p \vee (q \wedge \neg q] \vee q & \text{..... Distributive Law;} \\ & \equiv & [p \vee \mathbf{F}] \vee q & \\ & \equiv & p \vee q & \text{..... Identity Law} \end{array}$$
      This completes the proof.

   ii. $[(p \vee q) \Rightarrow r] \equiv [(p \rightarrow r) \wedge (q \Rightarrow r)]$.

      - **Ans:** The argument goes as follows:
      $$\begin{array}{rcll} (p \vee q) \Rightarrow r] & \equiv & \neg(p \vee q) \vee r & \text{..... Implication Law} \\ & \equiv & (\neg p \wedge \neg q) \vee r & \text{..... DeMorgan's Law} \\ & \equiv & (\neg p \vee r) \wedge (\neg q \vee r) & \text{..... Distributive Law} \\ & \equiv & (p \Rightarrow r) \wedge (q \Rightarrow r) & \text{..... Implication Laws} \end{array}$$

b) (5 points) Suppose X and Y are sets. Express the following as formulas involving quantifiers. Let $P(x)$: x is an element of X; $Q(x)$: x is an element of Y.

   i. Every element of X is an element of Y. **Ans:** $\forall\, x, (P(x) \wedge Q(x))$.

   ii. Some element of X is an element of Y. **Ans:** $\exists\, x, (P(x) \wedge Q(x))$.

   iii. Some element of X is not an element of Y.**Ans:** $\exists\, x, (P(x) \wedge \neg Q(x)$.

   iv. No element of X is an element of Y. **Ans:** $\neg(\forall\, x, (P(x) \wedge Q(x))$.

c) (5 points) Suppose $p(x)$ and $q(y)$ are open statements with the universe being real $\mathbb{R}$.

   i. Is $\forall\, x\, \forall\, y\ (p(x) \Rightarrow q(y))$ equivalent to $\forall\, x\ (p(x)) \Rightarrow \forall\, y\ (q(y))$? Justify.

      - **Ans:** This is not equivalent. This is seen from the following example. Let $p(x) : x = 2$, and let $q(y) : y = 3$. Now $\forall\, x\, \forall\, y\ (p(x) \Rightarrow q(y))$ is obviously false. Since $\forall x\ p(x)$ is false, $\forall\, x\ (p(x)) \Rightarrow \forall\, y\ (q(y))$ is true, by definition.

   ii. Is $\exists\, x\, \exists\, y\ (p(x) \wedge q(y))$ equivalent to $\exists\, x\ (p(x)) \wedge \exists\, y\ (q(y))$? Justify.

      - **Ans:** This is equivalent. Consider $x = c$ and $y = d$ where $p(c)$ and $q(d)$ are true. As a result of these, both $\exists\, x\, \exists\, y\ (p(x) \wedge q(y))$ and $\exists\, x\ (p(x)) \wedge \exists\, y\ (q(y))$ are true. They remain equivalent when such c or d does not exist.

3. Questions on Counting. Answer any two of the following questions. Clear explanations are needed.

   a) (10 points) How many elements are there in the set
   $$\{a | (a \in \mathbb{N}) \wedge (a \leq 1000) \wedge ((2|a) \vee (3|a) \vee (5|a))\}?$$

      - **Ans:** The question asks for the number of (positive) integers no more than 1000 which are divisible by 2, 3 or 5.

Let A_2, A_3, A_5 be the set of numbers (≤ 1000) divisible by 2, 3, 5 respectively. In this case $|A_2| = \lfloor \frac{1000}{2} \rfloor$; $|A_3| = \lfloor \frac{1000}{3} \rfloor$; $|A_5| = \lfloor \frac{1000}{5} \rfloor$. Moreover, let A_{23}, A_{25}, A_{35}, and A_{235} represent the sets whose elements are ≤ 1000 and are divisible by $2 \times 3, 2 \times 5, 3 \times 5, 2 \times 3 \times 5$ respectively. In this case $|A_{23}| = \lfloor \frac{1000}{6} \rfloor$, $|A_{25}| = \lfloor \frac{1000}{10} \rfloor$, $|A_{35}| = \lfloor \frac{1000}{15} \rfloor$, and $|A_{235}| = \lfloor \frac{1000}{30} \rfloor$.

The answer to the question is, therefore, $|A_2| + |A_3| + |A_5| - |A_{23}| - |A_{25}| - |A_{35}| + |A_{235}|$.

b) (10 points) Determine the number of times the following pseudocode prints the PRINT statement. Explain your answer.

```
for i = 1 to 20
   for j = i to 20
      for k = j to 20
         PRINT(i,j,k)
```

- **<u>Ans:</u>** Note here that $i \leq j \leq k$. Also note that given any three elements a, b, c, all are between 1 and 20, we can assign the smallest to i, the largest to k and the rest to j. This implies that the required number is the number of 3-combinations with repetitions from a set of 20 objects. Thus the answer is $C(20+3-1, 20-1)$ (20 kids, 3 pennies, and no constraints).

c) (10 points) Six students are to sit at a round table to discuss a project. John and Jean are among them, and they refuse to be seated next to each other. How many valid seating arrangements are possible?

- **<u>Ans:</u>** We fix the chair that John occupies. If there were no restrictions on others, the number of ways to seating the remaining 5 students is 5.4.3.2.1. However, the number of choices is reduced if Jean is not allowed to seat beside John. Therefore, the actual number is 4.3.3.2.1. The number of choices for the seats around John are 4 and 3.

d) (10 points) In how many ways can the letters in M, A, T, R, I, C, E, S be rearranged, if no two vowels are allowed to be next to each other?

- **<u>Ans:</u>** There are 3 vowels and 5 consonants in the word MATRICES. We first place 5 consonants in 5! ways. For each such permutation of consonants, we observe that there are 6 place to insert 3 vowels. There are $C(6,3)$ such choices. More over any permutation of 3 vowels in these positions realize a different arrangement. Therefore, the number of rearrangements possible with no vowels together is $5! \times C(6,3) \times 3!$.

e) (10 points) Determine the number of integral solutions to the following equation:

$$x_1 + x_2 + x_3 + x_4 \leq 20, \quad 1 \leq x_i, \ \forall\, i\, .$$

- **<u>Ans:</u>** First of all, the above problem is equivalent to the following problem (by replacing x_i by $y_i + 1$).

$$y_1 + y_2 + y_3 + y_4 \leq 16, \quad 0 \leq y_i, \ \forall\, i\, .$$

We have argued in the class that the above problem is equivalent to the following problem:

$y_1 + y_2 + y_3 + y_4 + \mathbf{z} = 16, \quad 0 \le y_i, \ \ \forall\, i$, and $z \ge 0$.

Therefore, the answer to the above question is $C(5+16-1, 5-1)$.

f) (10 points) How many four-digit integers have sum 10?

- **Ans:** We are assuming that the integers are all positive. If we allow negative integers as well, we just have to double the count.

  Consider any 4-digit integer 3212. If we add the digits, the total is 8. Thus 3212 is not a valid integer. However, 3412 is a valid integer. So is 9100, or 0910.

  The answer to the question is the same as distributing 10 pennies to 4 kids where a kid can get 0 penny, and cannot get more than 9 pennies. Therefore, we need to eliminate 4 solutions:
  $(10,0,0,0), (0,10,0,0), (0,0,10,0)$, and $(0,0,0,10)$.

  Thus the answer is $C(4+10-1, 4-1) - 4$.

4. Answer one of the following:

a) (5 points) If $n \in \mathbb{Z}$, then $5n^2 + 3n + 7$ is odd.

- **Ans:** The proof is by cases. The cases are when n is either odd, or even.

  Case 1: n is even

  In this case, $5n^2$ and $3n$ are even, and thus $5n^2 + 3n$ is even. Therefore, $5n^2 + 3n + 7$ is odd.

  Case 1: n is odd

  In this case $5n^2$ and $3n$ are odd, and thus $5n^2 + 3n$ is even. Therefore, $5n^2 + 3n + 7$ is odd.

  This completes the proof.

b) (5 points) Suppose a is an integer. If $5|2a$, then $5|a$.

- **Ans:** Suppose $5|2a$. By definition, we know that $2a = 5.t$ for some integer t. Since 5 doesn't divide 2, therefore 5 must divide a, given that a is an integer.

c) If $n \in \mathbb{Z}$, then $n^2 + 3n + 4$ is even.

- **Ans:** The proof is by cases. The cases are when n is either odd, or even.

  Case 1: n is even

  In this case, n^2 and $3n$ are even, and thus $n^2 + 3n$ is even. Therefore, $n^2 + 3n + 4$ is even.

  Case 1: n is odd

  In this case n^2 and $3n$ are odd, and thus $n^2 + 3n$ is even. Therefore, $n^2 + 3n + 4$ is even.

  This completes the proof.

d) Suppose a is an integer. If $7|4a$, then $7|a$.

- **Ans:** Suppose $7|4a$. By definition, we know that $4a = 7.t$ for some integer t. Since 7 doesn't divide 4, therefore 7 must divide a, given that a is an integer.

5. Bonus question. Answer any one of the following.

a) (10 points) Using the connectives $\neg$ and $\wedge$, write down a proposition P over the boolean variables p,q,r such that P is true only when p,q,r have truth values FFT or TFT.

Can such a P be constructed only using the connectives $\wedge$ and $\Rightarrow$?

- **Ans:** Clearly (discussed in the class) the proposition P satisfying the constraints is $(\neg p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge r)$. We can now remove $\vee$ connective using DeMorgan's Law as follows. $\neg\neg\{(\neg p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge r)\}$ which is equivalent to $\neg[\{\neg(p \wedge \neg q \wedge r) \wedge \neg(p \wedge \neg q \wedge r)\}]$.

  P canot be constructed only using the connectives $\wedge$ and $\Rightarrow$. Note that when p,q,r are all **True**, P should answer **False**. We are looking for equivalent proposition $s \Rightarrow t$ where s and t are compound propositions involving p,q,r and connective $\wedge$. Since p,q,r are all true, s and t must all be true. In this case $s \Rightarrow t$ is true, a contradiction.

b) (10 points) Show that the product of k consecutive positive integers is divisible by $k!$.

- **Ans:** Let $n, n+1, \ldots, n+k-1$ be k consecutive positive integers. Now

$$\begin{aligned} n(n+1).......(n+k-1) &= \frac{(n+k-1).....(n+2).(n+1).n.(n-1)....2.1}{(n-1)(n-2)....2.1} \\ &= \frac{(n+k-1)!}{(n-1)!} \end{aligned}$$

  We know that number of k-combinations of $n+k-1$ objects without repetitions is $C(n+k-1,k)$, which is $\frac{(n+k-1)!}{k!(n-1)!}$. Note that $C(n+k-1,k)$ is integral. This implies that $k!$ divides $\frac{(n+k-1)!}{(n-1)!}$, i.e. $k!$ divides the product $n(n+1)(n+2).......(n+k-1)$.

  This completes the proof.

c) (10 points) Determine the number of integral solutions to the following equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 50, \quad 1 \leq x_i, \ \forall\, i\ , ...(1)$$

where for each i, $1 \leq i \leq 5$, x_i is a multiple of 5.

- **Ans:** We first replace the integral equation with an equivalent integral equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 50, \quad 5 \leq x_i, \ \forall\, i\ ,.......... (2)$$

  where for each i, $1 \leq i \leq 5$, x_i is a multiple of 5.

We replace each x_i by $5y_i$ in the above equation, and obtain the following equivalent integral equation:

$y_1+y_2+y_3+y_4+y_5=10, \quad 1 \le y_i, \ \forall\, i$, ...(3)

We know how to solve (3). The number of integral solutions to (3) is $C(10-1, 5-1)$ which is the same as that of the equation (2).