Logic

Logic is a systematic way of thinking that allows us to deduce new information from old information. Logic is the common language all mathematicians use. It is a glue that holds a string of statements (arguments) together. Three important topics of logic will be covered.

1. The truth table tell us the exact meaning of the words ‘and,’ ‘or,’ ‘not’ etc.

2. The rules of inference provide a system that allows new information (statements) to be produced from known information.

3. The logical rules allows changing statements into (potentially more useful) statements with the same meaning.

<u>Statements</u>

A **statement** is a sentence that is definitely true or definitely false. There is no ambiguity in such statements. A statement is known as **proposition**.

The area of logic that deals with propositions/statements is called **propositional calculus** or **propositional logic**. It was first developed by Aristotle, more than 2300 years ago.

Examples

- The sun rises in the east

- If a circle has radius r, its area is πr^2.

- $\sqrt{2} \notin \mathbb{Z}$

- $\{0,1,2\} \cap \mathbb{N} = \Phi$

Here we pair sentences or expressions that are not statements with similar expressions that are statements.

NOT Statements:	Statements:
Add 5 to both sides.	Adding 5 to both sides of $x-5=37$ gives $x=42$.
$\mathbb{Z}$	$42 \in \mathbb{Z}$
42	42 is not a number.
What is the solution of $2x=84$?	The solution of $2x=84$ is 42.

We often use letters to stand for specific statements, just as letters are used to denote numerical variables. The conventional letters used for propositional variable are $p, q, r, \ldots$.

The following list contains some examples of propositions.

- p: For every integer $n > 1$, the number $2^n - 1$ is a prime.
- q: Every polynomial of degree n has at most n roots.
- r: The function $f(x) = x^2$ is continuous.
- s_1: $\{0,1,2\} \cap \mathbb{N} = \phi$.
- s_2: If an integer x is a multiple of 6, then x is even.

Statements/propositions are everywhere in Mathematics. Here are some famous statements.

p: For all numbers $a, b, c, n \in \mathbb{N}$ with $n > 2$, it is the case that $a^n + b^n \neq c^n$. (**Fermat's last theorem**)

q: Every even integer greater than 2 is a sum of two prime numbers. (**Goldbach conjecture**)

Try the problems in section 2.1 of the text.

Logical Connectives (Sections 2.2, 2.3, 2.4, 2.5)

Logical connectives are used to create a compound proposition from two or more propositions. The connectives we will user are listed below. p and q are assumed to be two statements.

1. AND (denoted by $\wedge$) or Logical conjunction
2. OR (denoted by $\vee$) or Logical disjunction (Inclusive Or)
3. XOR (denoted by $\oplus$) or Exclusive Or
4. Negation (denoted by $\neg$)
5. Conditional statements
   (a) Implication (denoted by $\Rightarrow$ (used in the text), $\rightarrow$ (used in the reference books)
   (b) Biconditional (denoted by $\Leftrightarrow$ (used in the text), $\leftrightarrow$ (used in the reference books)

These connectives can be summarized as follows.

Operator	Symbol	Usage
Conjunction	$\wedge$	and
Disjunction	$\vee$	or
Negation	$\neg$	not
Exclusive or	$\oplus$	xor
Conditional	$\Rightarrow$	if, then
Biconditional	$\Leftrightarrow$	if and only if (iff)

The following truth table illustrates these operations.

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$\neg p$	$\neg q$	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$
T	T	T	T	F	F	F	T	T	T
T	F	F	T	T	F	T	F	T	F
F	T	F	T	T	T	F	T	F	F
F	F	F	F	F	T	T	T	T	T

Consider the statements $p, q, r,$ and s as stated below.

p:	I finish writing my program before lunch.
q:	I shall play tennis in the afternoon.
r:	The sun is shinning.
s:	I shall take rest in the afternoon.

We can explain the logical connectives involving the above statements as follows.

(a) $\neg p$: I am not finishing writing my program before lunch.

(b) $p \wedge q$: I finish writing my program before lunch and I shall play tennis in the afternoon.
$p \wedge q$ is true when both p and q are true.

(c) $p \vee q$: I finish writing my program before lunch or I shall play tennis in the afternoon.
$p \vee q$ is true if one or the other of p and q is true or if both of the statements p and q are true. This 'or' ($\vee$) is also known as 'inclusive or'.

(d) $p \oplus s$: I finish writing my program before lunch or I shall take rest in the afternoon, but not both.
This or is called 'exclusive or'.

(e) $r \Rightarrow q$: If the sun is shinning, then I shall play tennis in the afternoon.
This statement is the implication of q by r. r is called the **hypothesis** of the implication and q is called the **conclusion**. Note that $r \Rightarrow q$ is still true if p is False.

We can restate $r \Rightarrow q$ as

- If r, then q.
- Whenever r, then also q.
- r is a sufficient condition for q.
- q is a necessary condition for r.

Consider statement $\underline{\text{You pass the exam} \Rightarrow \text{You pass the course}}$

- Passing the exam is sufficient for passing the course.
- For you to pass this course, it is sufficient that you pass this exam.

(f) $r \Leftrightarrow q$**:** The sun is shining if and only if I play tennis in the afternoon.
$r \Leftrightarrow q$ is logically equivalent to $(r \Rightarrow q) \wedge (q \Rightarrow r)$. The conditional statement $q \Rightarrow r$ is called the **converse** of $r \Rightarrow q$.

Example: Suppose p : Triangle ABC is isosceles, q : Triangle ABC is equilateral, r : Triangle ABC is equiangular.

Translate each of the following into an English sentence. Indicate also whether the statement is true.

(a) $q \Rightarrow p$ (b) $\neg p \Rightarrow \neg q$ (c) $q \Leftrightarrow r$ (d) $p \wedge \neg q$ (e) $r \Rightarrow p$.

A **truth table** is an important tool for evaluating the value of a compound proposition. This is a useful tool if the number of propositions is few. The truth table below illustrates the value of the proposition “The product $xy = 0$ if and only if $x = 0$ or $y = 0$” i.e. $(xy = 0) \Leftrightarrow ((x = 0) \vee (y = 0))$. Let $P : xy = 0$, $Q : x = 0$, $R : y = 0$ be the propositions.

P	Q	R	$Q \vee R$	$P \Leftrightarrow (Q \vee R)$
T	T	T	T	**T**
T	T	F	T	**T**
T	F	T	T	**T**
T	F	F	F	**F**
F	T	T	T	**F**
F	T	F	T	**F**
F	F	T	T	**F**
F	F	F	F	**T**

We have seen how to construct the truth table of any compound proposition.

Question: Is it possible to construct the proposition from a given truth table? The answer is yes, and this can be built in mechanical way.

Example:

Consider the following truth table where the proposition is not known. We are interested in determining the proposition.

p	q	?
T	T	F
T	F	T
F	T	T
F	F	F

The rule to determine the missing proposition is to visit each row whose entry in the last column is True. In the example, the second and third rows have entries T in the third column. This says that the missing proposition is true when (p=T and q=F) or (p=F and q=T). This means that the unknown proposition is true when $(p \wedge \neg q)$ or $(\neg p \wedge q)$ is true. This means that the unknown proposition is true when $(p \wedge \neg q) \vee (\neg p \wedge q)$ is true. Note that $(p \wedge \neg q) \vee (\neg p \wedge q)$ is false for the combinations of p and q in other rows (row 1 and 4) where the unknown proposition is known to be false. Therefore, the unknown proposition is $(p \wedge \neg q) \vee (\neg p \wedge q)$ whose truth value is last column.

Is there any other formula which is equivalent? It is possible.

The number of logical operations in the above case is five. Can we do with the smallest? It is a hard problem in general.

Try the problems in sections 2.2, 2.3, 2.4, 2.5

Logical Equivalence

Two statements are **logically equivalent** if their truth values match up line-for-line in a truth table. Logical equivalence is important because it gives us different ways of looking at the same thing. It is known that $p \Rightarrow q$, $\neg p \vee q$ and $\neg q \Rightarrow \neg p$ are equivalent. We can write $(p \Rightarrow q) = (\neg p \vee q) = (\neg q \Rightarrow \neg p)$. This is seen in the following truth table.

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$\neg p \vee q$	$\neg q \Rightarrow \neg p$
T	T	F	F	T	T	T
T	F	F	T	F	F	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

The fact that $p \Rightarrow q$ is equivalent to $\neg q \Rightarrow \neg p$ is very useful. In order to prove $p \Rightarrow q$, it may be easier to prove $\neg q \Rightarrow \neg p$.

A compound proposition that is always true no matter what the truth values of the involved propositions is called a **tautology**. A compound proposition that is always false is called a **contradiction**. $p \vee \neg p$ is a simple tautology. $p \wedge \neg p$ is a simple contradiction.

Informally, two compound propositions P and Q are equivalent if whenever P is true, Q is true; and whenever Q is true, P is true. In other words we can say that the proposition $P \Leftrightarrow Q$ is a tautology. The equivalence of P and Q is generally denoted by

- $P = Q$ (used in the text)
- $P \equiv Q$
- $P \leftrightarrow Q$
- $P \Leftrightarrow Q$ (used in the books by Grimaldi and Rosen)

Laws of Logic

Law of Double Negation	$\neg\neg p = p$	
Contrapositive Law	$p \Rightarrow q = (\neg q) \Rightarrow (\neg p)$	
DeMorgan's Law	$\neg(p \wedge q) = \neg p \vee \neg q$	$\neg(p \vee q) = \neg p \wedge \neg q$
Commutative Law	$p \wedge q = q \wedge p$	$p \vee q = q \vee p$
Distributive Law	$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$
Associative Law	$p \wedge (q \wedge r) = (p \wedge q) \wedge r$	$p \vee (q \vee r) = (p \vee q) \vee r$
Idempotent Law	$p \wedge p = p$	$p \vee p = p$
Identity Law	$p \wedge T = p$	$p \vee F = p$
Inverse law	$p \wedge \neg p = F$	$p \vee \neg p = T$
Domination Law	$p \wedge F = F$	$p \vee T = T$
Absorption Law	$p \wedge (p \vee q) = p$	$p \vee (p \wedge q) = p$

Some of the important equivalences involving conditional propositions are listed below.

- $p \Rightarrow q = \neg p \vee q$
- $p \Rightarrow q = \neg q \Rightarrow \neg p$
- $(p \Rightarrow q) \wedge (p \Rightarrow r) = (p \Rightarrow (q \wedge r))$
- $(p \Rightarrow r) \wedge (q \Rightarrow r) = ((p \vee q) \Rightarrow r)$
- $(p \Rightarrow q) \vee (p \Rightarrow r) = (p \Rightarrow (q \vee r))$
- $(p \Rightarrow r) \vee (q \Rightarrow r) = ((p \wedge q) \Rightarrow r)$
- $(p \Leftrightarrow q) = (p \Rightarrow q) \wedge (q \Rightarrow p)$
- $(p \Leftrightarrow q) = (\neg p \Leftrightarrow \neg q)$

An example:

Consider a piece of code: " *if ((x>0) and (y >0)) then writeln (....)*". Consider an equivalent piece of code: "*if (x > 0) then if (y > 0) then writeln (...)*". Why are they equivalent?

Let $p : x > 0$, $q : y > 0$, $r :$ a line is written. The above equivalent codes are implying that $(p \wedge q) \Rightarrow r$ is equivalent to $p \Rightarrow (q \Rightarrow r)$. How do you show that? We can show this through truth table. The following approach (better), using the laws of logic, is another way to show this.

1.		$(p \wedge q) \Rightarrow r$ **(l.h.s)**	
2.	$=$	$\neg(p \wedge q) \vee r$	Implication Law
3.	$=$	$(\neg p \vee \neg q) \vee r$	DeMorgan's Law
4.	$=$	$\neg p \vee (\neg q \vee r)$	Associative Law
5.	$=$	$\neg p \vee (q \Rightarrow r)$	Implication Law
6.	$=$	$p \Rightarrow (q \Rightarrow r)$ $(\mathbf{r.h.s})$	Implication Law

A general approach to proving equivalence using the laws of logic

- Remove double negation

- Remove implication by disjunction ($\vee$).
- Push negation inside the parentheses using DeMorgan's law.
- Use distribution law.

Try the problems in sections 2.6

Predicate Logic (Sections covered are 2.7, 2.8, 2.9, 2.10.)

The objective of this part is

- to learn the notational system for the predicate calculus.
- to learn the rules of logical equivalence for quantified statements.

Propositional statements are not powerful enough to capture wide range of statements. We will introduce a more powerful type of logic called predicate logic. Predicate logic is also known as the First Order Logic.

Statements using variables, such as

- $x+2$ is an even integer,
- y has four sides,
- x has black hair, etc.

are often used in various domains. These relations may or may not hold depending on the values of the variable x. When a sentence P contains a variable such as x, we sometimes denote it as $p(x)$ to indicate that it is saying something about x. A sentence whose truth depends on the value of one or more variables, is called an **open statements**. Here p is called a predicate or propositional function, and x is called an argument. Thus the above statements can be written as open statements as follows.

- $p(x)$: $x+2$ is an even integer. $p(2), p(15)$ are propositions.
- $p(y)$: y has four sides. $p(triangle), p(quadrilateral)$ are propositions.
- $p(x)$: x has black hair. $p(Nihar), p(Bishnu)$ are propositions.

Examples of open statements

- Mother(x): unary predicate
- Friend(x,y): binary predicate
- Sum(x,y,z): 3-ary predicate (Sum(x,y,z): x+y=z) (x,y,z) is a 3-tuple.
- $P(x_1, x_2, ..., x_n)$: n-ary predicates. $(x_1, x_2, ..., x_n)$ is a n-tuple.

Universe of Discourse:

Intuitively, **universe of discourse** is the set of values that can be assigned to a variable in an open statement (propositional function).

- $p(x)$: $x+2$ is an even integer. The universe of discourse is $\mathbb{Z}$. It could also be $\mathbb{N}$, or $\mathbb{R}$.
- $p(y)$: y has four sides. The universe of discourse is the set of polygons.
- $p(x)$: x has black hair. The universe of discourse is 'Polpulation of Canada'.
- Love(x,y): x loves y; The universe of discourse for x is SFU-Student-Group and for y is Flavored-Ice-Cream.

Quantifiers

We can use symbols $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow$ to deconstruct many English sentence into an equivalent symbolic form. This symbolic form allows us to understand the logical structure of the sentence. This way we can show how different sentences mean the same thing. However, these symbols are not enough.

Consider the statement, " For every $n \in \mathbb{Z}, 2n$ is even." If we write an open statement $p(x) : x$ is an even integer, the above statement is equivalent to $\ldots. \wedge p(2.(-2)) \wedge p(2.(-1)) \wedge p(2.0) \wedge p(2.1) \wedge p(2.2) \wedge \ldots.$ To help overcome this defect, two new symbols, called quatifiers, are introduces.

1. **Universal quantifier** $\forall$
   The proposition is true for all values in the universe of discourse.
   The above proposition then becomes "$\forall\, n \in \mathbb{Z}, 2n$ is even", or "$\forall\, n \in \mathbb{Z}, p(2n)$".

   The universal qualification of an open statement/proposition $p(x)$ is the proposition '**p(x) is true for all values of x in the universe of discourse**', which can be symbolically written as '$\forall$ **x** $\mathbf{p(x)}$'.

   If the universe of discourse is finite, say $\{a_1, a_2, \ldots, a_n\}$, then

$$\forall\, x \;\; p(x) \Leftrightarrow p(a_1) \wedge p(a_2) \wedge \ldots \wedge p(a_n)$$

2. **Existential quatifier** $\exists$
   The proposition is true for at least one value in the universe of discourse. Consider the sentence "There is an integer that is not even". The proposition can be written as "$\exists\, n \in \mathbb{Z}, \neg p(n)$".

   The existential quantification of an open statement/proposition $p(x)$ is the proposition '**there exists a value x in the universe of discourse such that p(x) is true**', which can be symbolically written as '$\exists$ **x p(x)**'.

   If the universe of discourse is finite, say $\{a_1, a_2, \ldots, a_n\}$, then

$$\exists\, x \;\; p(x) \Leftrightarrow p(a_1) \vee p(a_2) \vee \ldots \vee p(a_n)$$

Examples

1. Write following as English sentences:

   - $\exists\, m \in \mathbb{Z}, \forall\, n \in \mathbb{Z},\; m = n+5$
     This is equivalent to: There exists an integer m such that, for all n, $m = n+5$. This statement is false. The interpretation is left to right.
   - $\forall\, n \in \mathbb{N}, \exists\, X \subset \mathbb{Z}, |X| = n$.
     This is equivalent to: For any integer n, there exists a subset X of $\mathbb{Z}$ of size n.

2. Let p(x): x takes macm 101; q(x): x is a CS student. The universe is the student body at SFU.

   Express the statements:

   - Every CS student must take macm 101.
     $\forall x\, q(x) \rightarrow p(x)$.
   - There exists a non CS student who is taking macm 101.
     $\exists x(\neg q(x) \wedge p(x))$.
   - Everybody must take macm 101 or be a non CS student.
     $\forall x(p(x) \vee \neg q(x))$

3. Goldbach conjecture. Every integer greater than 2 is the sum of two primes.

   Let P denote the set of prime numbers. Let $E = \{4, 6, 8, ...\}$ be the set of all even numbers greater than 2.

$(n \in S) \Rightarrow (\exists p, q \in P,\ n = p + q)$,
or
$\forall n \in S, \exists p, q \in P,\ n = p + q$

Notice that $\forall x \in U,\ h(x)$ and $(x \in U) \Rightarrow h(x)$ are the same for any proposition $h(x)$ with universe U.

4. Write in symbol 'the equation $ax^2 + bx + c = 0$ has a real solution'.

   Let $p(x) : x = \frac{-b \pm \sqrt{b2 - 4ac}}{2a}$. The universe is $\mathbb{R}$.

   In symbol, $\exists x \in \mathbb{R}, p(x)$.

   Clearly, the statement is false. However, the statement is true if the universe in $\mathbb{C}$, the set of complex numbers.

Truth values of quantifiers.

The following table lists the truth table universal and existential quantifiers.

Statement	When True?	When False?
$\forall x\ p(x)$	$p(x)$ is true for every x	There exists an x for which $p(x)$ is false.
$\exists x\ p(x)$	There is an x for which $p(x)$ is true	$p(x)$ is false for every x

Try the problems in sections 2.7, 2.8, 2.9

Negating statements

We have seen that $\neg R$ is known as the negation of proposition R. DeMorgan's laws say:

$$\begin{array}{ccc} \neg(p \wedge q) & = & (\neg p) \vee (\neg q) \\ \neg(p \vee q) & = & (\neg p) \wedge (\neg q) \end{array}$$

We can use negation with quantified expressions the same way we used for propositions.

The following rules are:

- $\neg(\forall \mathbf{x}\ \mathbf{p}(\mathbf{x})) = \exists \mathbf{x}\ \neg \mathbf{p}(\mathbf{x})$
- $\neg(\exists \mathbf{x}\ \mathbf{p}(\mathbf{x})) = \forall \mathbf{x}\ \neg \mathbf{p}(\mathbf{x})$

The above rules are essentially are the qualified version of DeMorgan's laws when the universe of discourse is finite.

An example of negation

Rewrite: $\neg\forall x(\exists x\, \forall z\ p(x,y,x) \wedge\ \exists z\, \forall y\ p(x,y,z)$

Answer: $\exists x(\forall y\, \exists z\, \neg p(x,y,z) \vee \forall z\, \exists y\, \neg\ p(x,y,z))$

Try the problems in sections 2.10

Logical Inference

Automated theorem proving deals with the developments of computer programs that show that some statement is a logical consequence of a set of statements.

Suppose that a given statement $p \Rightarrow q$ is true, where p and q are propositions. This statement tells us that whenever p is true, q is true. However, it doesn't tell us whether p or q is true. We only can conclude that either both of them are false, or p is false and q is true. Suppose in addition we know that p is true. In this case we can conclude that q is true since $p \Rightarrow q$ is true. This conclusion is called logical inference. The following simple inferences are intuitively obvious. They represent certain patterns of reasoning that will be frequently used in proofs.

Rules of inference	Name
$p \Rightarrow q$ p $\therefore q$	Modus ponens
$p \Rightarrow q$ $\neg q$ $\therefore \neg p$	Modus tollens
$p \Rightarrow q$ $q \Rightarrow r$ $\therefore p \Rightarrow r$	Transitivity (hypothetical syllogism)
$p \vee q$ $\neg q$ $\therefore p$	Elimination (disjunctive syllogism)
$p \wedge q$ $\therefore p$	Simplification
p q $\therefore p \wedge q$	Conjunction
$\neg p \Rightarrow F$ $\therefore p$	Contradiction

Example: Determine whether the following argument is valid.

- She is a math major or a cs major.
- If she doesn't know discrete math, she is not a math major.
- If she knows discrete math, she is smart.
- She is not a cs major.

From the above hypotheses we can conclude that 'She is smart'. Why?

First of all, let us represent the above hypotheses in symbols. Consider the following propositions.

- p: She is a math major.
- q: She is a cs major.
- r: She knows discrete math.
- s: She is smart.

The hypotheses are:

1. $p \vee q$ is true.
2. $\neg r \Rightarrow \neg p$ is true
3. $r \Rightarrow s$ is true.
4. $\neg q$ is true.
5. $\therefore$ **s**

The inference logic goes as follows. Since $\neg q$ is true (#4), p is true, since #1 is true. Since p is true, r is true since, #2 is true. Therefore, s is true which follows from #3 since r is true.