Logic- Part-II

Some slides have been taken from the sites

http://cse.unl.edu/~choueiry/S13-235/

and

http://www.whitman.edu/mathematics/higher_math_online/section01.02.html

Predicate Logic (Propositional Function)

- Propositional statements are not powerful enough to capture wide range of statements.
- Consider the statement:

  For every n $\in$ Z, 2n is even

- Consider the sentences
  - y has four sides
  - x has black hair
  - x+2 is an even integer
- The above sentences involve variables.

Open sentences

- A sentence whose truth value depends on the value of one or more variables is called an open sentence.
- Examples of open statements
  - p(y): y has four sides
  - p(x): x has black hair
  - p(x): x+2 is an even integer
  - Mother(x): propositional function with one variable.
  - Friend(x,y): function with two variables. (2-tuple)
  - $P(x_1, x_2, \ldots, x_n)$: function with n variables. (n-tuple)
- An open statement p(x) is a proposition when x is assigned a value.

Universe of Discourse

- Intuitively, universe of discourse of a variable x in a propositional function is the set of values x can take.
- p(y): y has four sides.
  - Universe: set of polygons
- p(x): x has black hair
  - Universe: humans
- p(x): x + 2 is an even integer
  - Universe: set of integers

Quantifiers

- We can use ∧, ∨, ¬, ⇒, ⇔ to deconstruct many English sentence to an equivalent symbolic form.
- These symbols are not enough.
  - **For every n ∈ Z, 2n is even**

    Consider the open statement p(x): x is an even integer. Since the universe is Z, the above proposition can be written as

    [...∧ p(2.(-2)) ∧ p(2.(-1)) ∧ p(2(0)) ∧ p(2(1)) ∧ ...].

  – This is not much of help.

Universal Quantifier: $\forall$ (for all)

- $\forall x\ p(x)$: a proposition which is true if p(x) is true for all values of x of the universe of discourse.
- Consider : **For every n $\in$ Z, 2n is even**
  - Let p(x): x is even be an open statement.
  - **For every n $\in$ Z, 2n is even** can be wriiten as $\forall x \in Z, p(2x)$.
- If the Universe is finite, say $\{a_1, a_2, \ldots, a_n\}$, then

$$\forall x\ p(x) \Leftrightarrow p(a_1) \wedge p(a_2) \wedge \ldots \wedge p(a_n) .$$

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Examples : universal quantifiers

- Let p(x): x takes macm 101; q(x): x is a CS student.
- Express the statements:
  - Every CS student must take macm101

Examples : universal quantifiers

- Let $p(x)$: x takes macm 101; $q(x)$: x is a CS student.
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  - Every CS student must take macm101

$$\forall x\ (q(x) \Rightarrow p(x))$$

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  - Everybody must take macm101 or be a non CS student.

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- Express the statements:
  - Every CS student must take macm101

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  - Everybody must take macm101 or be a non CS student.

$$\forall x\ (p(x) \vee \neg q(x))$$

Examples : universal quantifiers

- Express the statement

  'for every x and every y, $x+y>10$'

- Answer:

  1. Let $P(x,y)$ be the statement $x+y>10$
  2. Where the universe of discourse for x, y is the set of integers
  3. The statement is: $\forall x \, \forall y \, P(x,y)$

- Shorthand: $\forall x,y \, P(x,y)$

Existential Quantifier: $\exists$ (there exists)

- $\exists x\, p(x)$: a proposition which is true if p(x) is true for at least one values of x of the universe of discourse.
- Consider : **There is an integer that is not even**
  - Let p(x): x is even be an open statement.
  - The above proposition can be wriiten as

    $$\exists x \in Z, \neg\, p(x).$$
- If the Universe is finite, say $\{a_1, a_2, \ldots, a_n\}$, then

$$\exists x\, p(x) \Leftrightarrow p(a_1) \vee p(a_2) \vee \ldots \vee p(a_n) .$$

Existential Quantifier: ∃ (there exists)

- Example: What is the truth value of $\exists x\ p(x)$ where $p(x)$ is the statement ``$x^2 > 10$”? Suppose the universe of discourse is set of positive number not exceedin 4, i.e. the set {1,2,3,4}

- Ans:

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- Example: What is the truth value of $\exists x \; p(x)$ where $p(x)$ is the statement ``$x^2 > 10$''? Suppose the universe of discourse is set of positive number not exceeding 4, i.e. the set {1,2,3,4}

- Ans:

  In our case $\exists x \; p(x) \Leftrightarrow p(1) \vee p(2) \vee p(3) \vee p(4)$.

  Since $p(4)$ is true, $\exists x \; p(x)$ is true.

Existential Quantifier: Examples

Let p(x): x takes macm 101; q(x): x is a CS student. The universe is the student body at SFU.

Express the statements:

- Every CS student must take macm 101.
  $\forall x\, q(x) \rightarrow p(x)$.
- There exists a non CS student who is taking macm 101.
  $\exists x(\neg q(x) \wedge p(x))$.
- Everybody must take macm 101 or be a non CS student.
  $\forall x(p(x) \vee \neg q(x))$

Quantifiers: Truth values

The truth table for the quantified statements.

Statement	When True?	When False?
$\forall x\ p(x)$	$p(x)$ is true for every x	There exists an x for which $p(x)$ is false.
$\exists x\ p(x)$	There is an x for which $p(x)$ is true	$p(x)$ is false for every x

Example:

- $p(x)$: $x > 0$; $q(x)$: $x^2 \geq 0$; $r(x)$: $x^2 - 3x - 4 = 0$
  $s(x)$: $x^2 - 3 > 0$.

Statement	Universe = real (+,-)	Universe: real (+)
$\exists x \ [p(x) \wedge q(x)]$		
$\forall x \ [p(x) \Rightarrow q(x)]$		
$\forall x \ [r(x) \vee s(x)]$		
$\forall x \ [r(x) \Rightarrow p(x)]$		

Example:

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$\exists x \ [p(x) \wedge q(x)]$	T	T
$\forall x \ [p(x) \Rightarrow q(x)]$	T	T
$\forall x \ [r(x) \vee s(x)]$	F	F
$\forall x \ [r(x) \Rightarrow p(x)]$	F	T

Precedence of Quantifiers

- The quatifiers ∀ and ∃ have higher precedence than all logical connectives.
  - $\forall x\ p(x) \wedge q(x)$ is the conjunction of $\forall x\ p(x)$ and $q(x)$.
  - it is equivalent to $(\forall x\ p(x)) \wedge q(x)$
  - but not equivalent to $\forall x\ (p(x) \wedge q(x))$

Problems

- Let $p(x,y)$: $xy = 0$; Universe is R.
  - $\forall x\ \forall y\ p(x,y) \Leftrightarrow \forall y\ \forall x\ p(x,y)$ (?)
  - $\forall x\ \exists y\ p(x,y) \Leftrightarrow \exists y\ \forall x\ p(x,y)$ (?)
- Let $q(x,y)$: $x+y = 0$; Universe is R
  - $\exists y\ \forall x\ q(x,y)$
  - $\forall x\ \exists y\ q(x,y)$

Problems

- Let p(x,y): xy =0; Universe is R.
  - $\forall x \, \forall y \; p(x,y) \Leftrightarrow \forall y \, \forall x \; p(x,y)$ (?) **(T)**
  - $\forall x \, \exists y \; p(x,y) \Leftrightarrow \exists y \, \forall x \; p(x,y)$ (?) **(T)**
- Let q(x,y): x+y = 0; Universe is R
  - $\exists y \, \forall x \; q(x,y)$ **(F)**
  - $\forall x \, \exists y \; q(x,y)$ **(T)**

Translating English to Symbolic Logic

- **Goldbach's conjecture:** Every even integer greater than 2 is the sum of two prime numbers.
  - Let P = { 2, 3, 5, 7, …} be the set of primes;
  - Let S= {4,6,8,10, …} be the set of even integers > 2.
  - $\forall x \in S, \exists p, q \in P, x = p + q$.

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- Every universally quantified statement can be expressed as a conditional statement.

Translating English to Symbolic Logic

- Every universally quantified statement can be expressed as a conditional statement.
  - The following statements mean the same thing.

    $\forall x \in S, Q(x)$

    $(x \in S) \Rightarrow Q(x)$

- Sometimes a theorem will be expressed as a universally quantified statement, but it will be more convenient to think of it as a conditional statement.

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We should be able to switch between the two forms

Translating English to Symbolic Logic

- If somebody is female and is a parent, this person is someone's mother
  - f(x): x is female
  - p(x): x is a parent
  - m(x,y): x is the mother of y

Translating English to Symbolic Logic

- If somebody is female and is a parent, this person is someone's mother
  - f(x): x is female
  - p(x): x is a parent
  - m(x,y): x is the mother of y

$$\forall x\ (f(x) \wedge p(x) \Rightarrow \exists y\ m(x,y))$$

Translating English to Symbolic Logic

- For every prime number p there is another prime number q with $q > p$.
  - $f(x)$: $x \in N$ is a prime.
  - $g(x,y)$: $x > y$, $x, y \in N$.

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- For every prime number p there is another prime number q with q > p.
  - f(x): $x \in N$ is a prime.
  - g(x,y): $x > y$, $x, y \in N$.

  - $\forall p \in N,\ f(x) \Longrightarrow \exists\, q \in N, f(q) \wedge g(q, p))$
  - $\forall p \in N, (f(x) \wedge (\exists\, q \in N, f(q) \wedge g(q, p)))$

Translating English to Symbolic Logic

- For every prime number p there is another prime number q with q > p.
  - f(x): $x \in$ N is a prime.
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Translating English to Symbolic Logic

P(x): x is a person; T(y): time is y; F(x,y): you can fool x in time y.

- You can fool some of the people all of the time

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  $\exists p\ (\forall t\ (P(p) \land (T(t)\ \Longrightarrow\ F(p,t))))$

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  $\exists p\ (\forall t\ (P(p) \wedge (T(t) \Rightarrow F(p,t))))$

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  $\exists t\ (\forall p\ (P(p) \wedge (T(t) \Rightarrow F(p,t))))$

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  $\forall p\ \forall t\ (P(p) \wedge (T(t)\ \Rightarrow F(p,t))))$

Negation

- We can use negation with quantified expressions as we used them with propositions
- Let $P(x)$ be a predicate. Then the followings hold:

$$\neg(\forall x\, P(x)) \equiv \exists x\, \neg P(x)$$

$$\neg(\exists x\, P(x)) \equiv \forall x\, \neg P(x)$$

- This is essentially the quantified version of De Morgan's Law.

Negation: Truth

Truth Values of Negated Quantifiers

Statement	True when…	False when…
$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$	$P(x)$ is false for every x	There is an x for which $P(x)$ is true
$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$	There is an x for which $P(x)$ is false	$P(x)$ is true for every x

Negation: Example

- Rewrite the following expression, pushing negation inward:

$$\neg \forall x\, (\exists y\, \forall z\, P(x,y,z) \wedge \exists z\, \forall y\, P(x,y,z))$$

- Answer:

$$\exists x\, (\forall y\, \exists z\, \neg P(x,y,z) \vee \forall z\, \exists y\, \neg P(x,y,z))$$

Example

- p(x): x is odd; q(x): $x^2 - 1$ is odd.
- $\forall x\ (p(x) \Rightarrow q(x))$
- $\neg\ [\forall x\ (p(x) \Rightarrow q(x))]$

$\Leftrightarrow\ \exists x\ [\ \neg(p(x) \Rightarrow q(x))]$

$\Leftrightarrow\ \exists x\ [\ \neg\ (\neg\ p(x) \vee q(x))]$

$\exists x\ [\ p(x) \wedge\ \neg\ q(x)\ ]$

Example

- p(x): x is odd; q(x): $x^2 - 1$ is odd.
- $\forall x\,(p(x) \Rightarrow q(x))$ True
- $\neg\,[\forall x\,(p(x) \Rightarrow q(x))]$

$\Leftrightarrow\ \exists x\,[\,\neg(p(x) \Rightarrow q(x))]$

$\Leftrightarrow\ \exists x\,[\,\neg\,(\neg\, p(x) \vee q(x))]$

$\exists x\,[\,p(x) \wedge \neg\, q(x)\,]$ False

Comments

- Whenever you see a quantifier, ask what is the universe of discourse.
- Should know the precedence rules. It is better to eliminate confusion by using the parentheses.
- The order of quantifiers matters a lot. Most often $\forall x\ \exists y\ q(x,y)$ is not equal to $\exists x\ \forall y\ q(x,y)$.
- DeMorgan's laws for quantifiers are very useful. It is important to be comfortable with DeMorgan's laws.
- You don't need to memorize the laws of logic. Just convince yourself that they are true.
- One way to show logical equivalence is through truth tables, at least when they do not have quantifiers over variables.

Practice problems from the text:

- Section 2.7
  - 1, 2, 4, 5, 7, 8
- Section 2.9
  - 5, 6, 8, 11
- Section 2.10
  - 3, 4, 5, 6

Some more practice problems (the universe is real)

Ex 1.2.1 Express the following as formulas involving quantifiers:

a) Any number raised to the fourth power is non-negative.

b) Some number raised to the third power is negative.

c) The sine of an angle is always between $+1$ and -1.

d) The secant of an angle is never strictly between $+1$ and -1.

Ex 1.2.2 Suppose X and Y are sets. Express the following as formulas involving quantifiers.

a) Every element of X is an element of Y.

b) Some element of X is an element of Y.

c) Some element of X is not an element of Y.

d) No element of X is an element of Y.

Some more practice problems (the universe is real)

Ex 1.2.3 Recall (from calculus) that a function f is **increasing** if

$$\forall a \forall b (a < b \Rightarrow f(a) < f(b))$$

Express the following definitions as formulas involving quantifiers:

a) f is **decreasing**.

b) f is **constant**.

c) f has a **zero**.

Ex 1.2.4 Express the following laws symbolically:

a) the commutative law of multiplication

b) the associative law of multiplication

c) the distributive law

Some more practice problems (the universe is real)

Ex 1.2.5 Are the following sentences true or false?

a) $\forall x \forall y (x < y \Rightarrow x^2 < y^2)$

b) $\forall x \forall y \forall z \neq 0 (xz = yz \Rightarrow x = y)$

c) $\exists x < 0 \exists y < 0 (x^2 + xy + y^2 = 3)$

d) $\exists x \exists y \exists z (x^2 + y^2 + z^2 = 2xy - 2 + 2z)$

Ex 1.2.6 Suppose $P(x)$ and $Q(y)$ are formulas.

a) Is $\forall x \forall y (P(x) \Rightarrow Q(y))$ equivalent to $\forall x (P(x)) \Rightarrow \forall y (Q(y))$?

b) Is $\exists x \exists y (P(x) \wedge Q(y))$ equivalent to $\exists x (P(x)) \wedge \exists y (Q(y))$?