Logic- Part-I

Some slides have been taken from the sites

http://cse.unl.edu/~choueiry/S13-235/

and

http://www.csee.umbc.edu/~ypeng/S03203/S03203.html

Logic

- Important for mathematical reasoning, program design.
- Used for designing electronic circuitry

- (Propositional) Logic is a system based on statements (also called propositions).

Logic

- A statement is a (declarative) sentence that is either true or false (not both).
- We say that the truth value of a proposition is either true (T) or false (F).
- Corresponds to 1 and 0 in digital circuits
- We usually denote a proposition by a letter: $p, q, r, s, \ldots$

- Consider a sentence: The sun rises in the east

- Is it a statement?

- What is the truth value
  of the proposition?

- Consider a sentence: The sun rises in the east
- Is it a statement? YES
- What is the truth value of the proposition? TRUE

- Consider a sentence: $\{0,2,3\} \cap N = \Phi$
- Is it a statement?
- What is the truth value of the proposition?

- Consider a sentence: $\{0,2,3\} \cap N = \Phi$
- Is it a statement? YES
- What is the truth value of the proposition? False

- Consider a sentence: $y > 21$
- Is it a statement?
- What is the truth value of the proposition?

- Consider a sentence: $y > 21$
- Is it a statement? No
- What is the truth value of the proposition? Its truth value depends on unspecified y. This statement is called an open statement.

- Consider a sentence: Please do not fall asleep.
- Is it a statement?
- What is the truth value of the proposition?

- Consider a sentence: Please do not fall asleep.
- Is it a statement? No
- What is the truth value of the proposition? It is neither true nor false.

Sentences that are not statements with similar expressions that are statements

NOT Statements:	Statements:
Add 5 to both sides.	Adding 5 to both sides of $x-5=37$ gives $x=42$.
$\mathbb{Z}$	$42 \in \mathbb{Z}$
42	42 is not a number.
What is the solution of $2x=84$?	The solution of $2x=84$ is 42.

- Consider a sentence:
  - Every even integer greater than 2 is a sum of two prime numbers.
  - Goldbach conjecture
- Is it a statement?

- What is the truth value of the proposition?

- Consider a sentence:
  - Every even integer greater than 2 is a sum of two prime numbers.
  - Goldbach conjecture
- Is it a statement? Yes
- What is the truth value of the proposition? Probably true

- Consider a sentence:
  - Either x is a multiple of 7 or it is not
- Is it a statement?

- What is the truth value of the proposition?

- Consider a sentence:
  - Either x is a multiple of 7 or it is not
- Is it a statement? Yes
- What is the truth value of the proposition? True since it is true for all x.

Logical Connectives (Operators)

- Combining statements to make compound statements.
- *p*, *q*, *r*, *s*, … represent statements/propositions.
- Following connectives are considered now.

Operator	Symbol	Usage
Conjunction	$\wedge$	and
Disjunction	$\vee$	or
Negation	$\neg$	not
Exclusive or	$\oplus$	xor
Conditional	$\Rightarrow$	if, then
Biconditional	$\Leftrightarrow$	if and only if (iff)

Logical Connective: Logical And

- The logical connective AND is true only when both of the propositions are true. It is also called a conjunction
- Examples
  - It is raining and it is warm
  - (2+3=5) and (1<2).
- Truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Logical Connective: Logical OR

- The logical disjunction, or logical OR, is true if one or both of the propositions are true.
- Examples
  - It is raining or it is the second lecture
  - (2+2=5) v (1<2)
- **Truth table**

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Logical Connective: Negation

- $\neg p$, the negation of a proposition p, is also a proposition
- p: Today is Monday
- Examples:
  - Today is not Monday
  - It is not the case that today is Monday, etc.
- **Truth table**

p	$\neg p$
T	F
F	T

Logical Connective: Exclusive Or

- The exclusive OR, or XOR, of two propositions is true when exactly one of the propositions is true and the other one is false
- Example
  - The circuit is either ON or OFF but not both
  - Let *ab*<0, then either *a*<0 or *b*<0 but not both
- Truth table

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Logical Connective: Implication (1)

- **Definition:** Let *p* and *q* be two propositions. The implication $p \rightarrow q$ is the proposition that is false when *p* is true and *q* is false and true otherwise
  - *p* is called the hypothesis, antecedent, premise
  - *q* is called the conclusion, consequence
- **Truth table**

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Logical Connective: Implication (2)

- The implication of $p \rightarrow q$ can be also read as
  - *p* implies *q*
  - If *p,* then *q*
  - whenever *p*, then also *q*
  - *q* follows from *p*
  - p only if q (p cannot be true if q is not true)
  - *p* is a sufficient condition for *q* (*p* is sufficient for *q*)
  - *q* is a necessary condition for *p* (*q* is necessary for *p*) (p cannot be true unless q is true (i.e. if q is false, p is false)

Logical Connective: Implication ()

- Consider the statements:
  - you pass the exam → you pass the course

- Equivalent statements:
  - Passing the exam is sufficient for passing the course.
  - For you to pass this course, it is sufficient that you pass the exam.

Exercise: Which of the following implications is true?

- If -1 is a positive number, then 2+2=5

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  True. Same as above.

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- If you get an 100% on your Midterm 1, then you will have an A^+.

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- If you get an 100% on your Midterm 1, then you will have an A^+.

  False. Your grades homework, quizzes, Midterm 2, and Final, if they are bad, would prevent you from having an A^+.

Exercises

- To take discrete mathematics, you must have taken calculus or a course in computer science.
- When you buy a new car from Acme Motor Company, you get $2000 back in cash or a 2% car loan.
- School is closed if more than 2 feet of snow falls and if the wind chill is below -80.

Exercises

- To take discrete mathematics, you must have taken calculus or a course in computer science.
- Propositions
  - p: take discrete math
  - q: you have taken calculus
  - r : you have taken a course in CS
- $p \Rightarrow q \vee r$

Exercises

- When you buy a new car from Acme Motor Company, you get $2000 back in cash or a 2% car loan.
- Propositions
  - p: you buy a car
  - q: you you get $2000 back
  - r : you get 2% car loan
- $p \Rightarrow q \oplus r$

Exercises

- School is closed if more than 2 feet of snow falls and if the wind chill is below -80.
- Propositions
  - p: School is closed
  - q: 2 feet of snow falls
  - r : wind chill is below -80

$$q \wedge r \Rightarrow p$$

Logical Connective: Biconditional (1)

- **Definition:** The biconditional $p \Leftrightarrow q$ is the proposition that is true when *p* and *q* have the same truth values. It is false otherwise.
- Note that it is equivalent to $(p \Rightarrow q) \wedge (q \Rightarrow p)$
- **Truth table**

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Logical Connective: Biconditional (2)

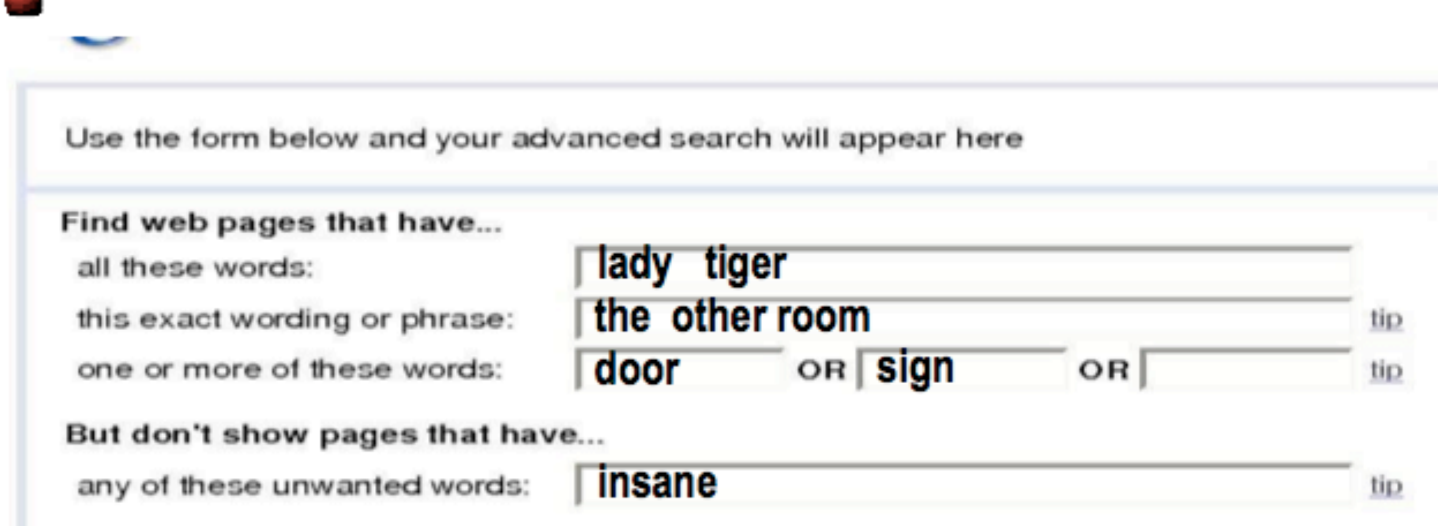
- The biconditional $p \leftrightarrow q$ can be equivalently read as
  - p if and only if q
  - p is a necessary and sufficient condition for q
  - if p then q, and conversely
  - p iff q
- Examples
  - $x>0$ if and only if x^2 is positive
  - The alarm goes off iff a burglar breaks in

Truth Tables

- Truth tables are used to show/define the relationships between the truth values of
  - the individual propositions and
  - the compound propositions based on them

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$\neg p$	$\neg q$	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$
T	T	T	T	F	F	F	T	T	T
T	F	F	T	T	F	T	F	T	F
F	T	F	T	T	T	F	T	F	F
F	F	F	F	F	T	T	T	T	T

Web Search



- (lady ∧ tiger) ∧ (the other room) ∧ (door ∨ sign) ∧ ¬ insane

Precedence of Logical Operators

- As in arithmetic, an ordering is imposed on the use of logical operators in compound propositions
- However, it is preferable to use parentheses to disambiguate operators and facilitate readability

$$\neg p \vee q \wedge \neg r \equiv (\neg p) \vee (q \wedge (\neg r))$$

- To avoid unnecessary parenthesis, the following precedence hold:

1. Negation ($\neg$) —— **highest**
2. Conjunction ($\wedge$)
3. Disjunction ($\vee$); Exclusive-or ($\oplus$)
4. Implication ($\Rightarrow$)
5. Biconditional ($\Leftrightarrow$) ———**lowest**

Examples

- Sentence ?
  - Some sets are finite.
  - N $\in$ PowerSet (N)
- Express each statement or open statement in one of the forms: p $\wedge$ q, p $\vee$ q, or $\neg$p.
  - Today is cold but it is not cloudy
  - x $\in$ A – B
  - At least one of the numbers x and y equals 0.

Examples

- Express the following sentence in the form **If P, then Q.**
  - We will order pizza today if there are 100 students in the class.
  - (If there are 100 students, then we will order pizza today)
  - We will order pizza today only if there are 100 students in the class.
  - (If we order pizza today, there are 100 students in the class)
  - You can use the lab if you are a cs major or not a freshman.
  - (if you are a cs major or not a freshman, then you can use the lab.)
  - An integer is divisible by 8 only if it is divisible by 4.
  - (If an integer is divisible by 8, then it is divisible by 4.)

Logical Equivalence

- Two statements are logically equivalent if their truth values match up line-for-line in a truth table.
- Logical equivalence gives us different ways of looking at the same thing.
- Consider the following truth table concerning $p \Rightarrow q$.

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$\neg p \vee q$	$\neg q \Rightarrow \neg p$
T	T	F	F	T	T	T
T	F	F	T	F	F	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Example

- If m is an even integer then m+7 is odd.
- Let p: m is an even integer; q: m+7 is odd.
- Using propositional notation p $\Longrightarrow$ q.
- Proving p $\Longrightarrow$ q.
  - p is true

$\Longrightarrow$ m=2a, a $\in$ Z

$\Longrightarrow$ 2a + 6 is an even number

$\Longrightarrow$ (2a +6) +1 is an odd number

$\Longrightarrow$ m + 7 is an odd number

$\Longrightarrow$ q

Example

- Proving $\neg q \Rightarrow \neg p$.
  - q is not true

$\Longrightarrow$ m+7 is even

$\Longrightarrow$ $m + 7 = 2b, b \in Z$

$\Longrightarrow$ $2b - 7 = 2(b-4) + 1$

$\Longrightarrow$ $2b - 7$ is odd

$\Longrightarrow$ m is odd

$\Longrightarrow$ $\neg p$

Tautology

- A compound proposition (a formula) that is always true no matter what the truth values of the propositions in the formula is called a **tautology**.
  - $p \vee \neg p$ is a simple tautology.
  - $p \Longrightarrow q$ and $\neg q \Longrightarrow \neg p$ are logically equivalent

    i.e. $(p \Longrightarrow q) \Longleftrightarrow (\neg q \Longrightarrow \neg p)$ is a tautology.

    (Truth table will be shown in the class)

Tautology

- A compound proposition (a formula) that is always true no matter what the truth values of the propositions in the formula is called a **tautology**.
- Informally, two compound propositions P and Q are logically equivalent if whenever P is true, Q is true, and whenever Q is true then P is true.
  – In other words $P \Leftrightarrow Q$ is a tautology
  – It is denoted as $P = Q$ (in the text)
  – Also denoted as $P \equiv Q$ (Rosen); $P \Leftrightarrow Q$ (Grimaldi)

Contradiction

- A compound proposition (a formula) that is always false no matter what the truth values of the propositions in the formula is called a **contradiction**.
  - $p \land \neg p$ is a simple contradiction.

The following table contains some important equivalences

Laws of Logic

Law of Double Negation	$\neg\neg p \Leftrightarrow p$	
Contrapositive Law	$p \Rightarrow q \Leftrightarrow (\neg q) \Rightarrow (\neg p)$	
DeMorgan's Law	$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
Commutative Law	$p \wedge q \Leftrightarrow q \wedge p$	$p \vee q \Leftrightarrow q \vee p$
Distributive Law	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

The following table contains some important equivalences (contd.)

Associative Law	$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
Idempotent Law	$p \wedge p \Leftrightarrow p$	$p \vee p \Leftrightarrow p$
Identity Law	$p \wedge T \Leftrightarrow p$	$p \vee F \Leftrightarrow p$
Inverse law	$p \wedge \neg p \Leftrightarrow F$	$p \vee \neg p \Leftrightarrow T$
Domination Law	$p \wedge F \Leftrightarrow F$	$p \vee T \Leftrightarrow T$
Absorption Law	$p \wedge (p \vee q) \Leftrightarrow p$	$p \vee (p \wedge q) \Leftrightarrow p$

Some equivalences involving conditional statements

Some of the important equivalences involving conditional propositions are listed below.

- $p \Rightarrow q = \neg p \vee q$
- $p \Rightarrow q = \neg q \Rightarrow \neg p$
- $(p \Rightarrow q) \wedge (p \Rightarrow r) = (p \Rightarrow (q \wedge r))$
- $(p \Rightarrow r) \wedge (q \Rightarrow r) = ((p \vee q) \Rightarrow r)$
- $(p \Rightarrow q) \vee (p \Rightarrow r) = (p \Rightarrow (q \vee r))$
- $(p \Rightarrow r) \vee (q \Rightarrow r) = ((p \wedge q) \Rightarrow r)$
- $(p \Leftrightarrow q) = (p \Rightarrow q) \wedge (q \Rightarrow p)$
- $(p \Leftrightarrow q) = (\neg p \Leftrightarrow \neg q)$

Example

- if ((x > 0) and (y > 0)) then writeln(...) is equivalent to if (x > 0) then if (y > 0) then writeln (---)
- Propositions (x and y are constants)
  - p : x > 0
  - q : y > 0
  - r : line gets written
- Claim is $(p \wedge q \Rightarrow r) = (p \Rightarrow (q \Rightarrow r))$

Example

- if ((x > 0) and (y > 0)) then writeln(...) is equivalent to if (x > 0) then if (y > 0) then writeln (---)
- Propositions (x and y are constants)
  - p : x > 0
  - q : y > 0
  - r : line gets written
- Claim is (p ∧ *q* ⇒ *r)* = (p ⇒ (q ⇒ r))

1.		$(p \wedge q) \Rightarrow r$ **(l.h.s)**	
2.	$\equiv$	$\neg(p \wedge q) \vee r$	Implication Law
3.	$\equiv$	$(\neg p \vee \neg q) \vee r$	DeMorgan's Law
4.	$\equiv$	$\neg p \vee (\neg q \vee r)$	Associative Law
5.	$\equiv$	$\neg p \vee (q \Rightarrow r)$	Implication Law
6.	$\equiv$	$p \Rightarrow (q \Rightarrow r)$ **(r.h.s)**	Implication Law

A general approach to proving equivalence using the laws of logic

- Remove double negation
- Remove implication by disjunction (v)
- Push negation inside the parentheses using DeMorgan's law
- Use distribution law

Other Examples

- Find a proposition with truth value

p	q	?
T	T	F
T	F	T
F	T	T
F	F	F

Other Examples

- Determine a logically equivalent proposition to $p \oplus q$.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

- Logically equivalent: $(p \wedge \neg q) \vee (\neg p \wedge q)$
- Replacing each row with value T with an equivalent proposition. These rows are then combined with $\vee$.

Using the truth table, one can show that …

- Any compound proposition (a formula) can be transformed to a logically equivalent formula involving only ¬, ∧, ∨ operators.

Other Examples

- Write a proposition equivalent to $\neg p \land \neg q$ using only $\neg$ and $\lor$.
- Determine whether $p \Rightarrow (\neg q \land r)$, $\neg p \lor \neg (r \Rightarrow q)$ are logically equivalent
- Prove that $(q \land (p \Rightarrow \neg q)) \Rightarrow \neg p$ is a tautology using propositional equivalence and laws of logic.
- Write the negation of
  - If it is sunny, we go skiing.
  - I will go the movie or read a book, but not both.

Practice problems from the text:

- Section 2.1
  - 2, 3, 10, 11
- Section 2.2
  - 1, 3, 4, 6, 7, 12
- Section 2.3
  - 1, 3, 9, 10, 11
- Section 2.4
  - 3, 4, 5
- Section 2.5
  - 2, 6, 7, 10, 11
- Section 2.6
  - 1, 2, 7, 9, 10, 12, 13