Induction (Chapter 10)

Some slides have been taken from the sites

http://cse.unl.edu/~choueiry/S13-235/

Motivation

- How can we prove the following proposition?

$$\forall x \in A\ S(x)$$

- For a finite set $A=\{a_1,a_2,\ldots,a_n\}$, we can prove that $S(x)$ holds for each element because of the equivalence

$$S(a_1) \wedge S(a_2) \wedge \ldots \wedge S(a_n)$$

- For an infinite set, we can try to use <u>universal generalization</u>
- Another, more sophisticated way is to use <u>*induction*</u>

Principle of Mathematical Induction

- To prove that a statement $S(n)$ is true for all positive integers n (i.e. $n \in N$) , we perform two steps:

  Basis step: We verify that $S(1)$ is true.

  Inductive step: We show that the conditional statement

  $$S(k) \Rightarrow S(k+1) \text{ is true } \forall k \in N$$

- Symbolically, the statement

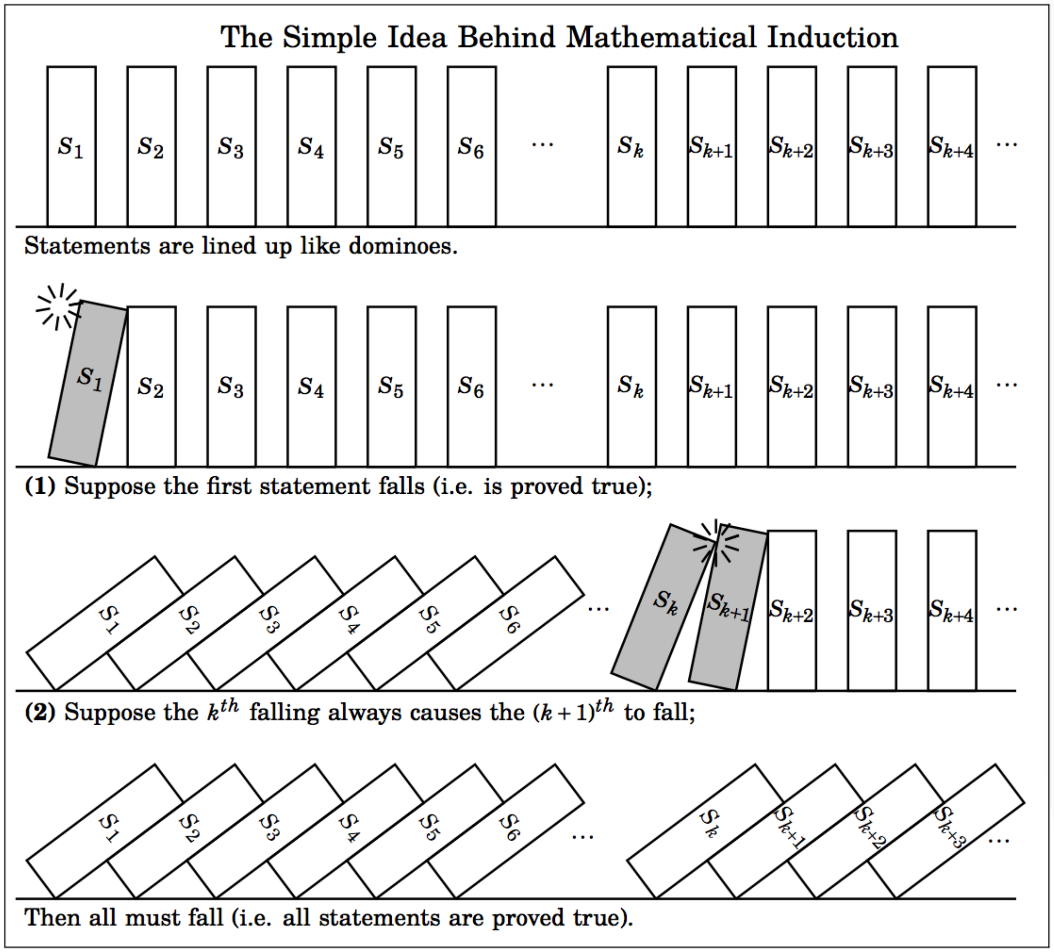
  $$(S(1) \wedge \forall(k \geq 1)\ (S(k) \Rightarrow S(k+1))) \Rightarrow \forall(n \in N)\ S(n)$$

Principle of Mathematical Induction

- How do we do this?
  - S(1) is usually an easy property to show.
  - To prove the conditional statement, we assume that S(k) is true (it is called inductive hypothesis) and show that under this condition S(k+1) is also true.

Mathematical Induction

- It is a powerful proof techniques.



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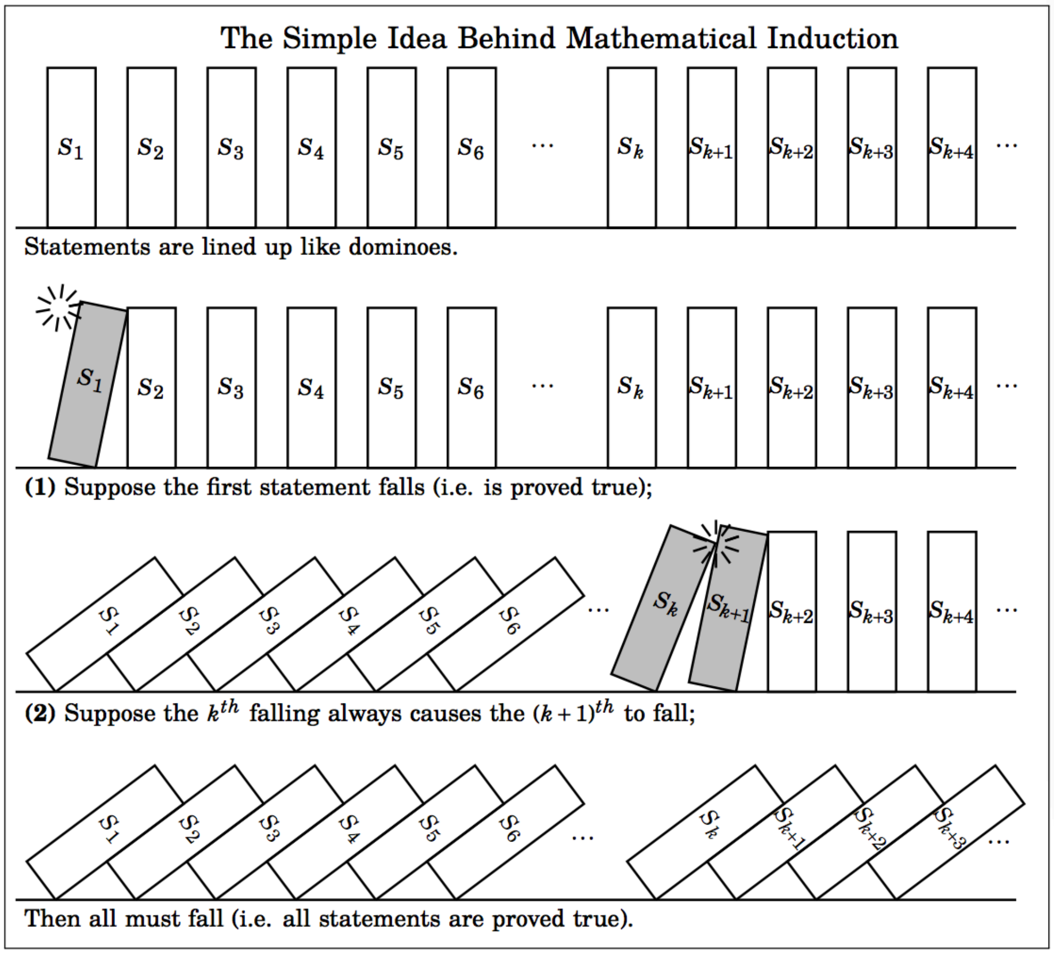

The Domino Effect

Show that all dominos fall.

- Basis Step: The first domino falls.
- Inductive step: Whenever a domino falls, its next neighbor will also fall.

Outline for Proof by Induction

Proposition The statements $S_1, S_2, S_3, S_4, \ldots$ are all true.

Proof. (Induction)

(1) Prove that the first statement S_1 is true.

(2) Given any integer $k \geq 1$, prove that the statement $S_k \Rightarrow S_{k+1}$ is true.

It follows by mathematical induction that every S_n is true. ■

Another View

- To look at it in another way, assume that the statements
  - S(1)
  - S(k) $\Rightarrow$ S(k+1)

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- Otherwise, we apply (2) above to get

  S(1) ⇒ S(2), S(2)⇒ S(3), S(3) ⇒ S(4), …, S(c-1) ⇒ S(c)

  via a finite number of steps (c-1) we get that S(c) is true.

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- Because c is arbitrary, the universal generalization is established and

  $$\forall n \in N \;\; S(n)$$

Example 1

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- We must prove that S(k+1) holds. That is, $2^{2(k+1)}-1$ is divisible by 3.

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- Note that: $2^{2(k+1)} - 1 = 2^2 2^{2k} - 1 = 4.2^{2k} - 1$
- The inductive hypothesis: $2^{2k} - 1 = 3t \Rightarrow 2^{2k} = 3t+1$
- Thus: $2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1 = 4(3t+1) - 1$

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- We conclude, by the principle of mathematical induction, for any integer $n \geq 1$, $2^{2n}-1$ is divisible by 3.

Example 2 (page 154)

Proposition If $n \in \mathbb{N}$, then $1+3+5+7+\cdots+(2n-1)=n^2$.

Open Statement S_n : $1 + 3 + 5 + \ldots\ldots + (2n-1) = n^2$.

Proof. We will prove this with mathematical induction.

(1) Observe that if $n = 1$, this statement is $1 = 1^2$, which is obviously true.

(2) We must now prove $S_k \Rightarrow S_{k+1}$ for any $k \geq 1$. That is, we must show that if $1+3+5+7+\cdots+(2k-1)=k^2$, then $1+3+5+7+\cdots+(2(k+1)-1)=(k+1)^2$.

- **The rest of the details will be shown in the class.**

Example 3: Summation

- **Show that $\Sigma_{i=1}^{n} (i^3) = (\Sigma_{i=1}^{n} i)^2$ for all $n \geq 1$.**
- The basis case is trivial: for n =1, $1^3 = 1^2$
- The inductive hypothesis assumes that for some n≥1 we have $\Sigma_{i=1}^{k} (i^3) = (\Sigma_{i=1}^{k} i)^2$
- We now consider the summation for (k+1): $\Sigma_{i=1}^{k+1} (i^3)$

$= (\Sigma_{i=1}^{k} i)^2 + (k+1)^3 \quad = (k(k+1)/2)^2 + (k+1)^3$

$= (k^2(k+1)^2 + 4(k+1)^3) /2^2 \quad = (k+1)^2 (k^2 + 4(k+1)) /2^2$

$= (k+1)^2 (k^2 +4k+4) /2^2 \quad = (k+1)^2 (k+2)^2 /2^2$

$= ((k+1)(k+2) / 2)^2$

- Thus, by the PMI, the equality holds

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Examples solved in the text

Proposition If n is a non-negative integer, then $5 \mid (n^5 - n)$.

Proposition If $n \in \mathbb{Z}$ and $n \geq 0$, then $\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$.

Proposition For each $n \in \mathbb{N}$, it follows that $2^n \leq 2^{n+1} - 2^{n-1} - 1$.

Proposition If $n \in \mathbb{N}$, then $(1+x)^n \geq 1 + nx$ for all $x \in \mathbb{R}$ with $x > -1$.

Example

Example: Suppose $a_1, a_2, \ldots, a_n$ are n integers, where $n \geq 2$. If p is prime and $p|(a_1 \times a_2 \times \ldots \times a_n)$, then $p|a_i$ for at least one of the a_i.

Proof: The proof is on induction on n.

- Let $S(n) : p|(a_1 \text{ x } a_2 \text{ x } \ldots \text{ x } a_n), n \geq 2$, p a prime integer.
- The basis step involves proving $p|(a_1 \text{ x } a_2)$.

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  - Suppose p does not divide a_1.
  - $\gcd(p, a_1) = 1$.

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  - Therefore, $1 = p.k + a_1 l$, k and l are elements of Z.

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  - $gcd(p,a_1) = 1$.
  - Therefore, $1 = p.k + a_1 l$, k and l are elements of Z.
  - We can write $a_2 = p.k.\ a_2 + a_1 l.\ a_2$
  - Since $p|(a_1 \text{ x } a_2)$, $p|a_2$.

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  - Suppose p does not divide a_1.
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  - We can write $a_2 = p.k.\ a_2 + a_1 l.\ a_2$
  - Since $p|(a_1 \text{ x } a_2)$, $p|a_2$.
  - Similar arguments for the case when p does not divide a_2.

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Proof: The proof is on induction on n.

- The basis step involves $n = 2$. Suppose $p|a_1a_2$. We have seen that either $p|a_1$ or $p|a_2$.
- Suppose that $k \geq 2$ and $p|(a_1 \times a_2 \times \ldots \times a_k)$ implies then $p|a_i$ for some a_i. (Inductive hypothesis)
- Now let $p|(a_1 \times a_2 \times \ldots \times a_k \times a_{k+1})$. Then $p|((a_1 \times a_2 \times \ldots \times a_k) \times a_{k+1})$. By what we proved in the basis step, it follows that $p|(a_1 \times a_2 \times \ldots \times a_k)$ or $p|a_{k+1}$. This and the inductive hypothesis imply that p divides one of the a_i.

Why Induction Works? Well Ordering

- One of the axioms of positive integers is the principle of well-ordering:

  Every non-empty subset of N contains the least element.

- Note that the sets of all integers, rational numbers, and real numbers do not have this property.

- Suppose that mathematical induction is not valid.

  Then there is a predicate $P(n)$ such that $P(1)$ is true,

  $\forall k\ (P(k) \to P(k+1))$ is true, but there is n such that $P(n)$ is false

  Let $T \subseteq N$ be the set of all n such that $P(n)$ is false.

  By the principle of well-ordering T contains the least element a

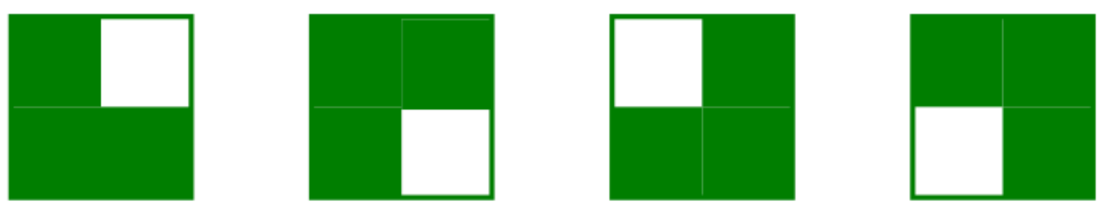
  As $P(1)$ is true, $a \neq 1$.

  We have $P(a-1)$ is true. However, since $P(a-1) \to P(a)$, we get a contradiction

Triomino

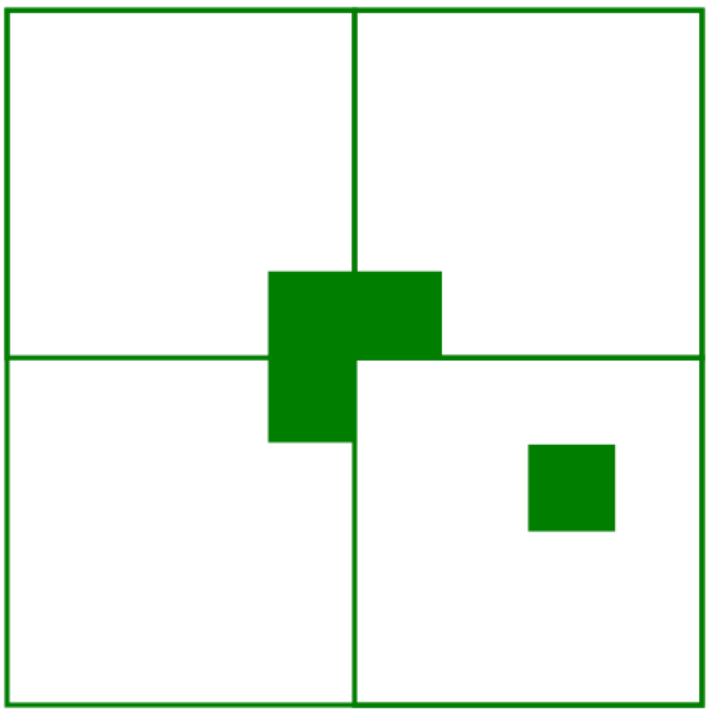
- Let n be a positive integer. Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using triominos

- P(n) denotes the statement above
- Basis step: P(1) is true, as 2×2 checkerboards with one square removed have one of the following shapes



Triomino (cntd)

- Inductive step: Suppose that P(k) is true that is every $2^k \times 2^k$ checkerboard with one square removed can be tiled with triominos.
  We have to prove P(k + 1), that is, every $2^{k+1} \times 2^{k+1}$ checkerboard without one square can be tiled.



Split the big checkerboard into 4 half-size checkerboards

Put one triomino as shown in the picture.

We have 4 $2^k \times 2^k$ checkerboards, each without one square. By the induction hypothesis, they can be tiled.

Example

- Show that any number larger than 43 can be written as the sum of nonnegative multiples of 6, 9, or 20.
  - i.e $n = 6\ t_1 + 9\ t_2 + 20\ t_3$, where $t_1, t_2, t_3 \geq 0$, $n \geq 44$.

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- Consider n=1,2,3,….
- We notice that only the numbers 6, 9, 12, 15, 18, 20, 21, 24, 26, 27, 29, 30, 32, 33, 36, 38, 39, 40, 42 (less than 43) can be expressed as the sum of 6 and 9.
- However, 43 is not expressible as the sum of 6, 9, and 20.

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- Consider n=1,2,3,….
- We notice that only the numbers 6, 9, 12, 15, 18, 20, 21, 24, 26, 27, 29, 30, 32, 33, 36, 38, 39, 40, 42 (less than 43) can be expressed as the sum of 6 and 9.
- However, 43 is not expressible as the sum of 6, 9, and 20.
- Claim any number greater than or equal to 44 can be expressed as
  - $S(n)$: $n = 6 t_1 + 9 t_2 + 20 t_3$, where $t_1, t_2, t_3 \geq 0$, $n \geq 44$.

Example (contd.)

- We can show that S(44), S(45), S(46), S(47), S(48), S(49) are true.
- We now show that

$$(S(44) \wedge S(45) \wedge .... \wedge S(k)) \Rightarrow S(k + 1),\ k \geq 49.$$

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- We now show that

  $$(S(44) \wedge S(45) \wedge .... \wedge S(k)) \Rightarrow S(k+1),\ k \geq 49.$$
- Consider m= (k+1)-6.
- Since k ≥ 49, m ≥ 44.
- We assumed (induction hypothesis) that S(m) is true.
- i.e. $m = 6\,t'_1 + 9\,t'_2 + 20\,t'_3$, where $t'_1, t'_2, t'_3 \geq 0$.
- Hence $m+6 = k+1 = 6\,(t'_1+1) + 9\,t'_2 + 20\,t'_3$.
- Therefore, S(k+1) is true. This means that
  $S(44) \Rightarrow S(50)$, $S(45) \Rightarrow S(51)$, $S(46) \Rightarrow S(52)$, $S(47) \Rightarrow S(53)$, …,
  completing the induction.

- By the principle of strong induction, $\forall n \in N\ S(n),\ n \geq 44$ is true.

Principle of Strong Mathematical Induction

- Sometimes mathematical induction is not enough.
- In order to prove that $S(n)$ is true for all positive integer $n \geq n_0$, we complete two steps:
  - Basis step: We verify that $S(n_0)$, $S(n_0+1)$, ..., $S(n_1)$ are true.
  - Inductive step: We show that conditional statement $(S(n_0) \wedge S(n_0+1) \wedge \ldots \wedge S(k)) \Rightarrow S(k + 1)$ for all positive integers $k \geq n_1$.
  - It follows by the principle of strong mathematical induction that every $S(n)$, $n \geq n_0$ is true.
- Symbolically

$$[(S(n_0) \wedge S(n_0+1) \wedge \ldots \wedge S(n_1) \wedge S(n_1+1) \wedge \ldots \wedge S(k) \Rightarrow S(k+1),\ k \geq n_1] \Rightarrow \forall (n \geq n_0)\ S(n)$$

Example

- Any integer $n > 11$ can be written in the form $n=4a + 5b$ for $a, b \in Z$.
- Try at home.

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6. Always finish your proof with something like: S(k+1) is true when S(k) is true, S(k-1) is true, etc., and therefore, S(n) is true for all n ≥ base case.

Using strong induction to graphs.

- A **graph** is a configuration consisting of points (called **vertices**) and **edges** which are lines connecting the vertices.
- Examples of graphs:

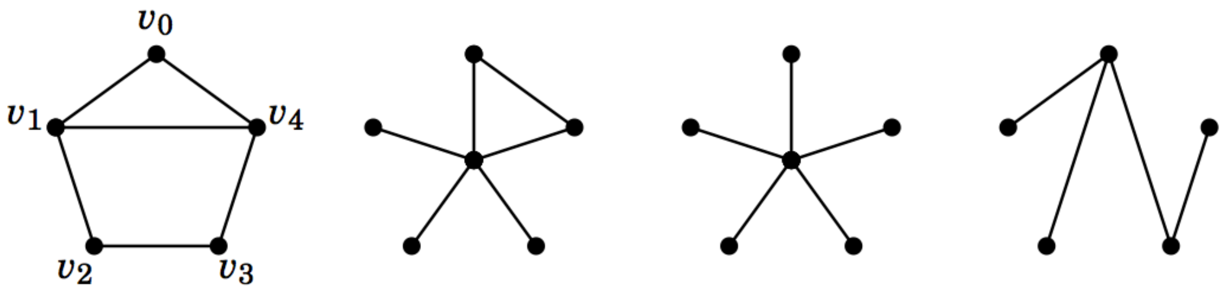

Figure 10.1. Examples of Graphs

- First two instances of the graphs have **cycles**, set of edges together forming a cycle.
- A graph with no cycles is called a **tree.** The two graphs on the right are trees.

Trees

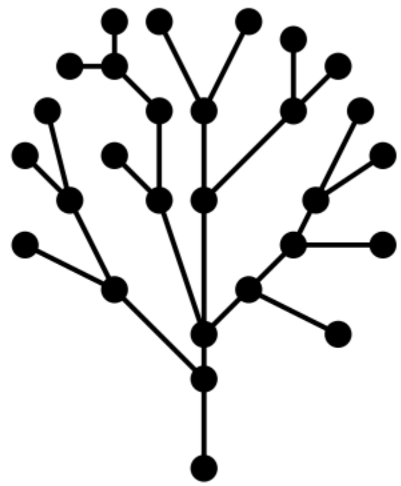

Figure 10.2. A tree

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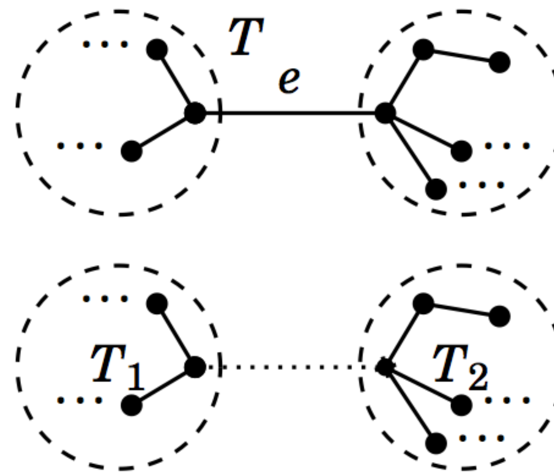
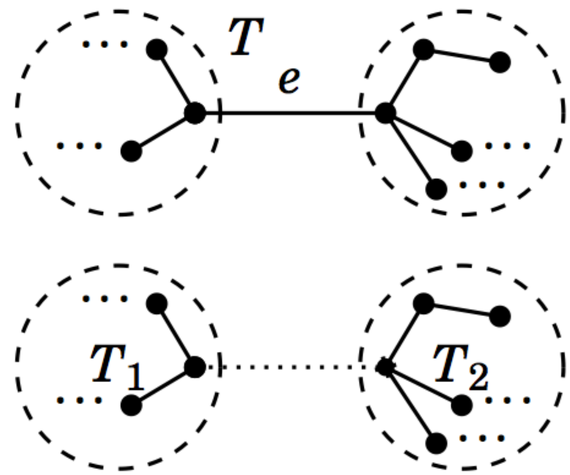


Showing S(k+1) is true.

- Consider an arbitrary tree T with k+1 vertices.
- The tree has at least one edge.
- Single out an edge e of T .
- Remove the edge e from T.
- This results in two smaller trees

  T_1 and T_2 with no more than k vertices.



Showing S(k+1) is true.

- Consider an arbitrary tree T with k+1 vertices.
- The tree has at least one edge.
- Single out an edge e of T .
- Remove the edge e from T.
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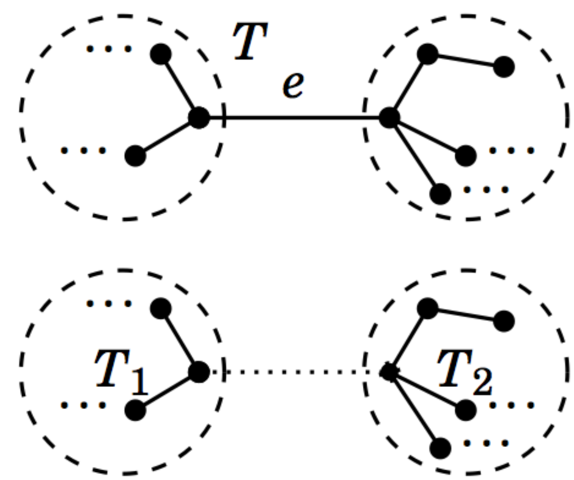
- Let T_1 has x vertices, and therefore, T_2 has (k+1)-x vertices.



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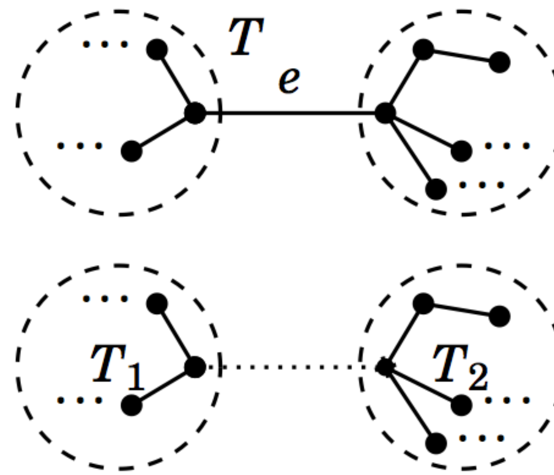

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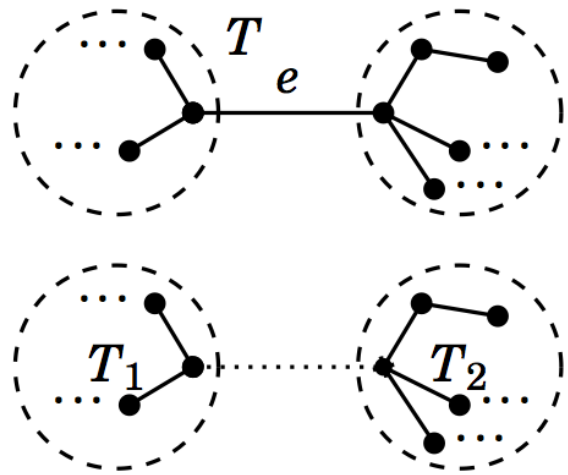

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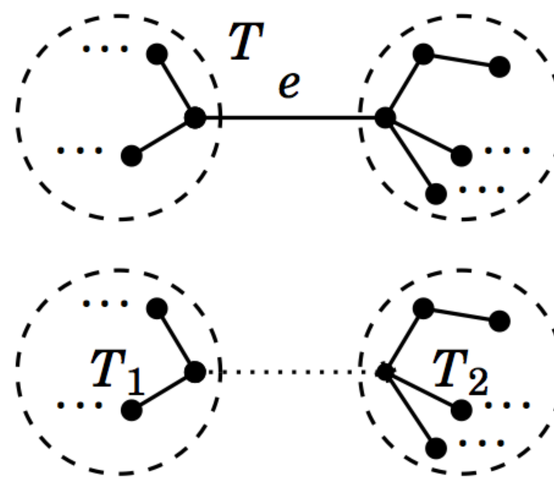

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- The total number of edges in T is (x-1)+(k-x) +1 = k. The edge e contributes one to the total. Hence S(k+1) is true.
- Follows by strong induction that a tree with n vertices has (n-1) edges.

Fibonacci Number

The *Fibonacci sequence* is defined as the sequence starting with $F_1 = 1$ and $F_2 = 1$, and then recursively as $F_n = F_{n-1} + F_{n-2}$.

The Fibonacci sequence starts off with 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Fibonacci Number

Proposition: Prove that

$$F_1 + F_2 + \ldots + F_n = F_{n+2} - 1 \text{ for } n \geq 2$$

Proof.

We use induction. As our basis step, notice that $F_1 + F_2 = F_4 - 1$ since

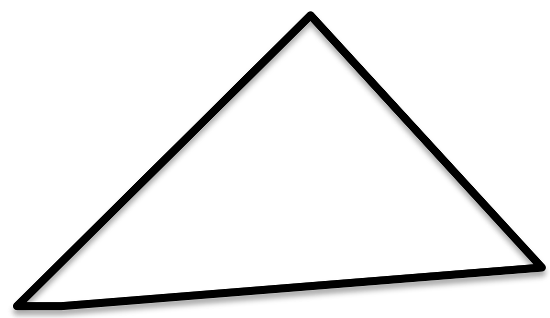
$$F_1 + F_2 = 1 + 1 = 2, \qquad \text{and} \qquad F_4 - 1 = 3 - 1 = 2.$$

Suppose that $F_1 + F_2 + \cdots + F_k = F_{k+2} - 1$. Adding F_{k+1} on both sides gives

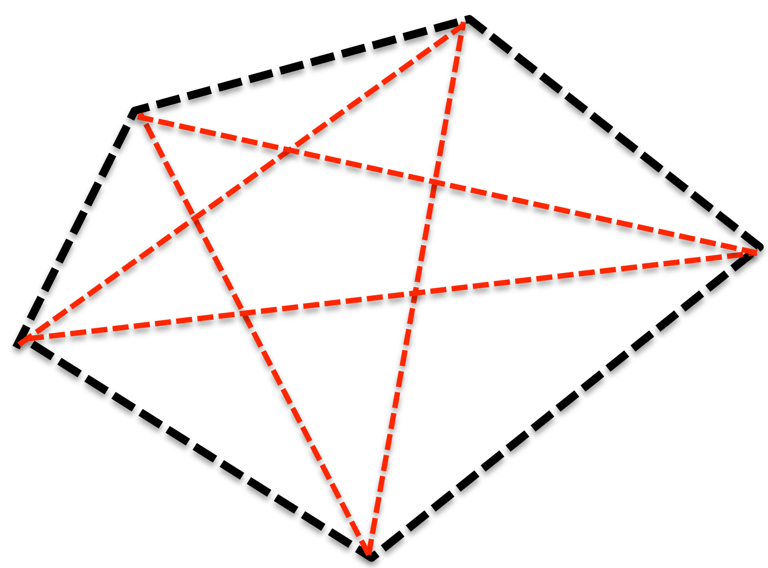
$$\begin{aligned} F_1 + F_2 + \cdots + F_k + F_{k+1} &= F_{k+2} - 1 + F_{k+1} \\ &= F_{k+1} + F_{k+2} - 1 \\ &= F_{k+3} - 1. \end{aligned}$$

which completes the induction. □

Number of diagonals in a convex polygon with n vertices.



n = 3; # of diagonal =0



n = 5; # of diagonal = 5

Number of diagonals in a convex polygon with n vertices is n(n-3)/2.

- S(n) : $n \geq 3$ vertex convex polygon realizes $N(n) = n(n-3)/2$ diagonals.
- Proof by Induction.
- Basis: S(3) is true, since a triangle realizes $N(3) = 0$ diagonal.
- Inductive Hypothesis: S(k) is true for any $k \geq 3$, $N(k) = k(k-3)/2$.
- Show that S(k+1) is also true.
  - Consider a convex polygon P with k+1 vertices $p_1, p_2, \ldots, p_{k+1}$.
  - $N(k+1) = N(k) + ((k-2) + 1)$
    $= k(k-3)/2 + k-1$
    $= (k+1)((k+1)-3)/2$ (**details in the class**)

- Using the principle of strong induction, $\forall\, n \geq 3$, S(n) is true.

Summation

- Prove that the sum of the first n natural numbers equals $\frac{n(n+1)}{2}$
  that is $$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

- P(n): `the sum of the first n natural numbers …
- Basis step: P(1) means $1=\frac{1(1+1)}{2}$

- Inductive step: Make the inductive hypothesis, P(k) is true, i.e.
  $$1+2+3+\cdots+k=\frac{k(k+1)}{2}$$
  Prove P(k + 1): $1+2+3+\cdots+k+(k+1)=\frac{(k+1)((k+1)+1)}{2}$
  $$1+2+3+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)$$
  $$=\frac{k(k+1)+2(k+1)}{2}=\frac{(k+1)(k+2)}{2}$$

More Summation

- Prove that $1+2+2^2+\cdots+2^n = 2^{n+1}-1$
- Let P(n) be the statement `$1+2+2^2+\cdots+2^n = 2^{n+1}-1$' for the integer n

- Basis step: P(0) is true, as $2^0 = 1 = 2^{0+1}-1$
- Inductive step: We assume the inductive hypothesis

$$1+2+2^2+\cdots+2^k = 2^{k+1}-1$$

and prove P(k + 1), that is

$$1+2+2^2+\cdots+2^k+2^{k+1} = 2^{(k+1)+1}-1 = 2^{k+2}-1$$

We have $1+2+2^2+\cdots+2^k+2^{k+1} = (2^{k+1}-1)+2^{k+1}$

$$= 2\cdot 2^{k+1}-1$$

$$= 2^{k+2}-1$$

The Cardinality of the Power Set

- We have proved that, for any finite set A, it is true that $|P(A)| = 2^{|A|}$
- Let Q(n) denote the statement `an n-element set has 2^n subsets'
- Basis step: Q(0), and empty set has only one subset, empty
- Inductive step. We make the inductive hypothesis, a k-element set A has 2^k subsets

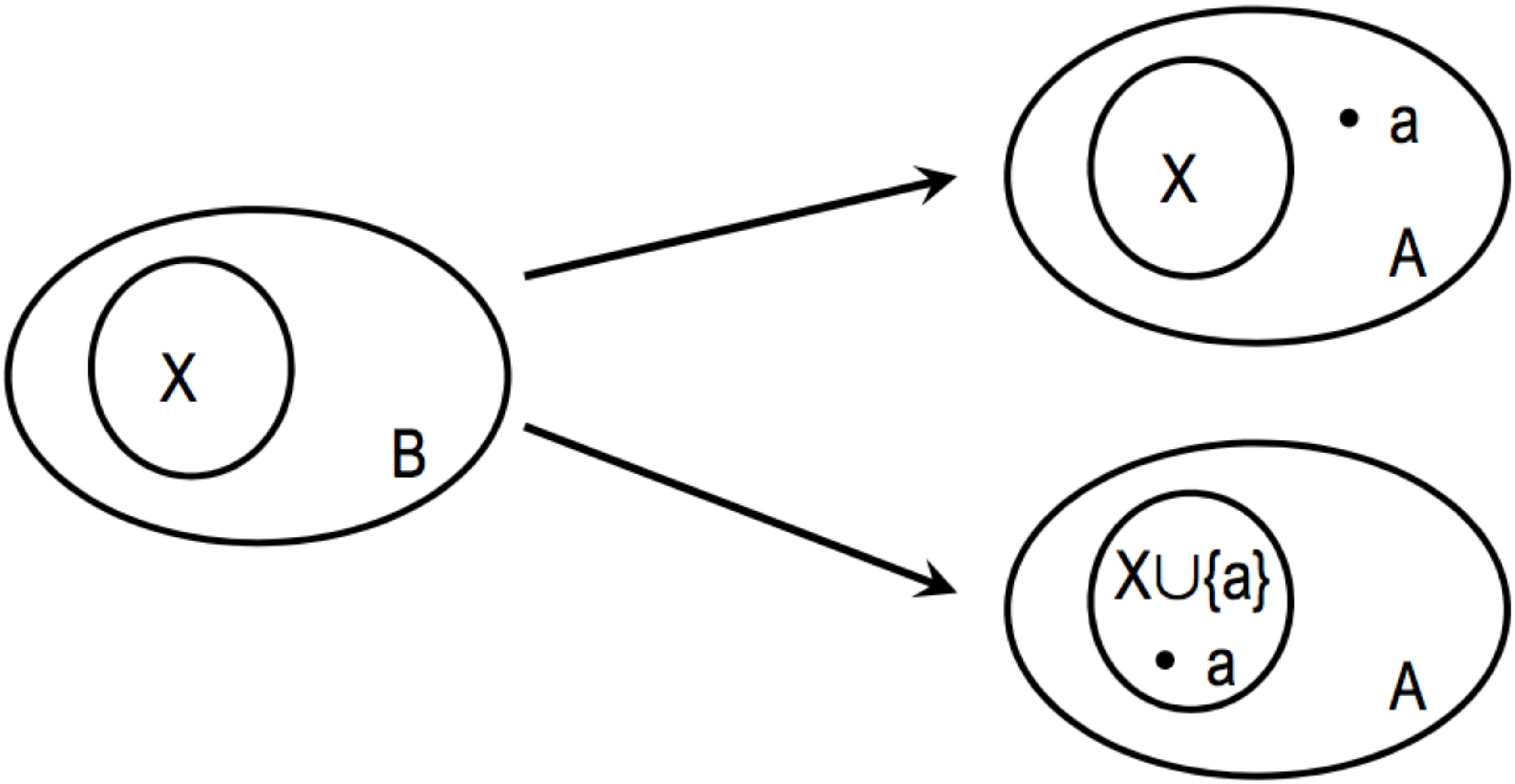
  We have to prove Q(k + 1), that is if a set A contains k + 1 elements, then $|P(A)| = 2^{k+1}$

  Fix an element $a \in A$, and set $B = A - \{a\}$.

  The set B contains k elements, hence $|P(B)| = 2^k$

  Every subset X of B corresponds to two subsets of A

The Cardinality of the Power Set (cntd)



Therefore, $|P(A)| = 2 \cdot |P(B)| = 2 \cdot 2^k = 2^{k+1}$

Practice problems from the text. (Chapter 10)

- 1,2,4,5,9, 11, 16, 23, 24, 33, 35