Functions

Chapter 12

Acknowledgement

- I have used materials from the following sources:
  - Textbook
  - Lecture notes slides prepared by Prof. Bulatov.
  - http://cse.unl.edu/~choueiry/S13-235/
  - http://www.math-cs.gordon.edu/courses/mat231/notes.html

Outline

- Functions
- Injective (one-to-one) and Surjective (onto) Functions
- Composition of Functions
- Inverse Functions
- The Pigeonhole Principle.

Functions

- The concept of relations between the sets plays a big role here.
- Consider the function $f(x) = x^2$, where $x \in R$.

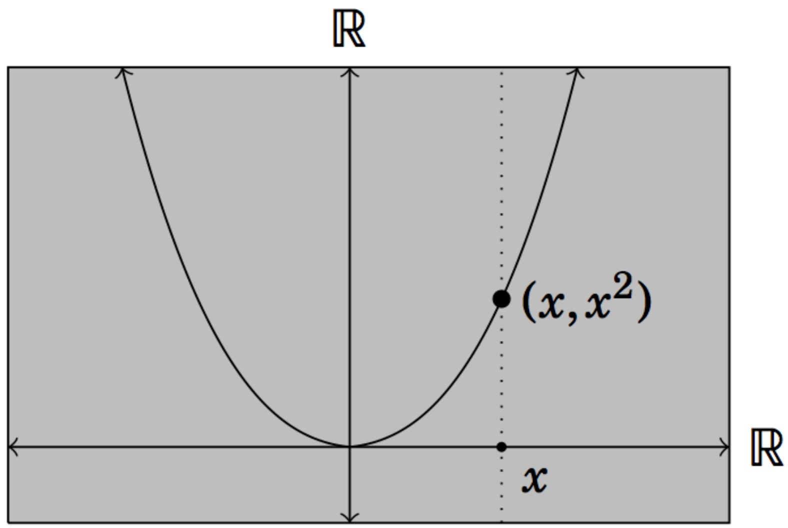


Figure 12.1. A familiar function

Functions

- The concept of relations between the sets plays a big role here.
- Consider the function $f(x) = x^2$, where $x \in R$.
- The points on the curve are related.
  - $T=\{(x,x^2): x \in R\} \subseteq R \times R$.
- Functions are special kinds of relations.

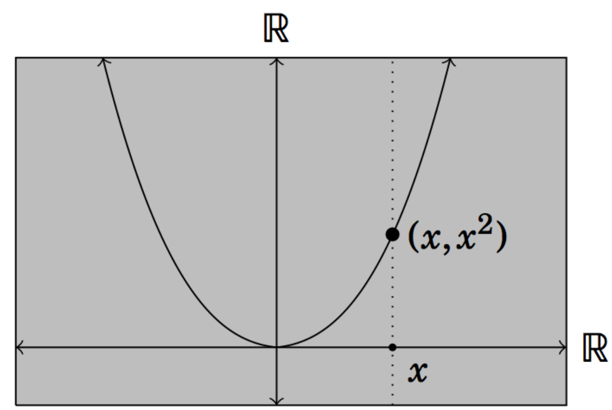


Figure 12.1. A familiar function

Another example of a function

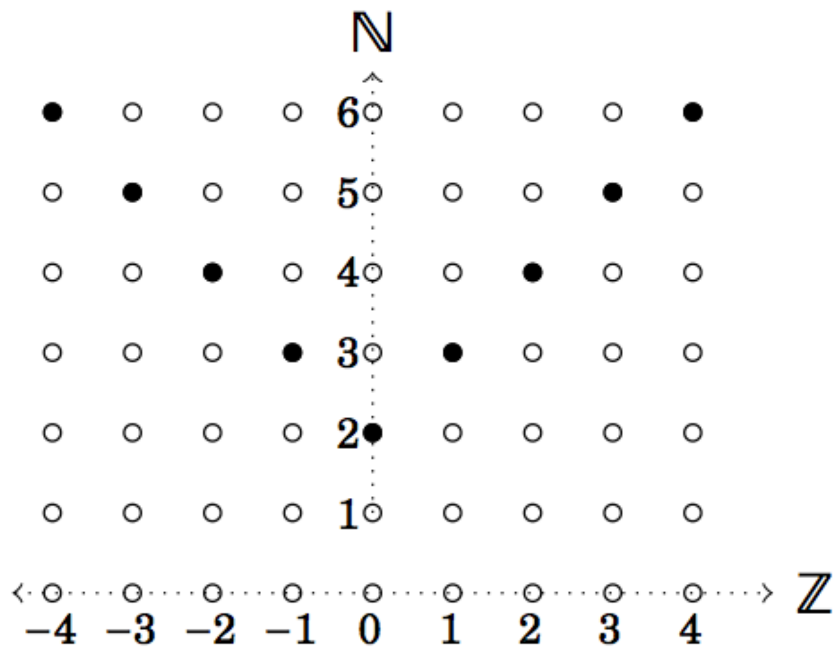


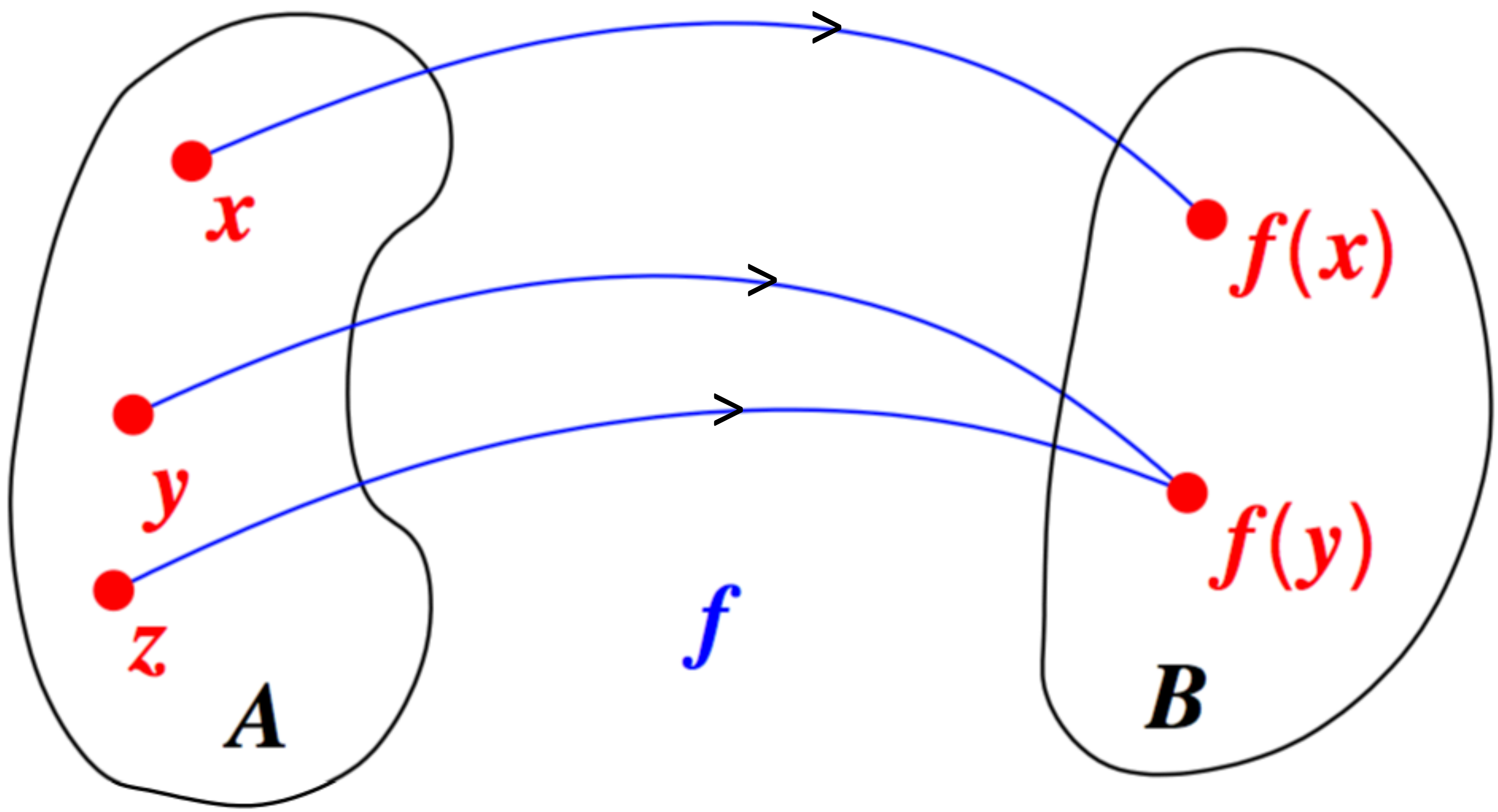
Figure 12.2. The function $f : \mathbb{Z} \to \mathbb{N}$, where $f(n) = |n| + 2$

- Here R= $\{(n,|n|+2): n \in N\} \subseteq Z \times N$.

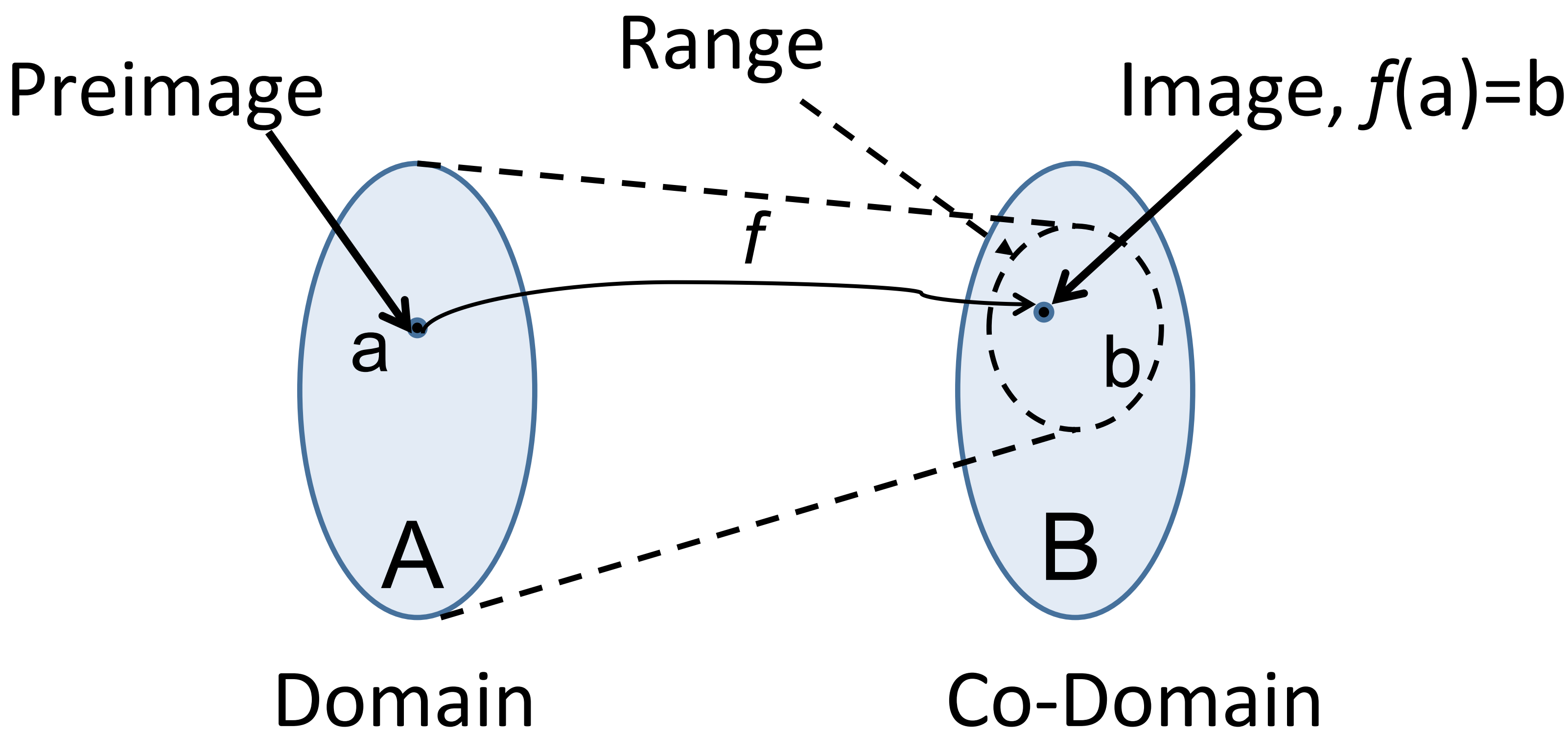
Functions

- Definition:
  - Suppose A and B are sets.
  - A function f from A to B (denoted as f: : $A \rightarrow B$) is a relation $f \subseteq A \times B$ from A to B.
  - The relation satisfies the property that for each element $a \in A$ the relation f contains exactly one ordered pair (2-tuple) of the form (a,b).
  - The statement $(a,b) \in f$ is abbreviated as $f(a) = b$.
- A shorter form of the definition
  - A function f from A to B is an assignment of exactly one element of B to each element of A.

We can diagram f: A → B as



Function: Visualization



A function, f: A $\rightarrow$ B

Terminology

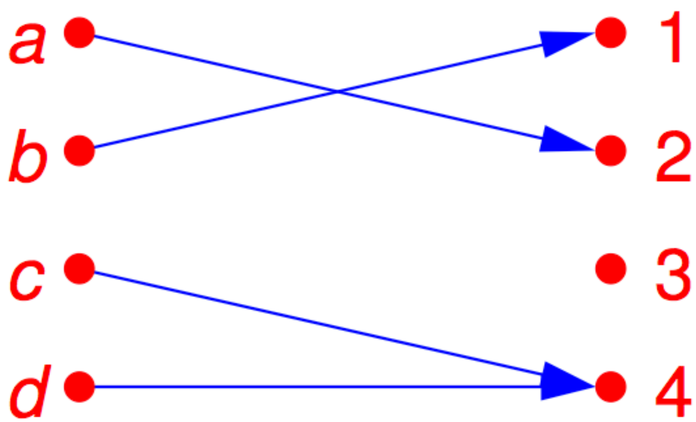
- Let f: A $\rightarrow$ B and f(a)=b. Then we use the following terminology:
  - A is the domain of f
  - B is the co-domain of f
  - b is the image of a
  - a is the preimage (antecedent) of b
  - The range of f is the set of all images of elements of A

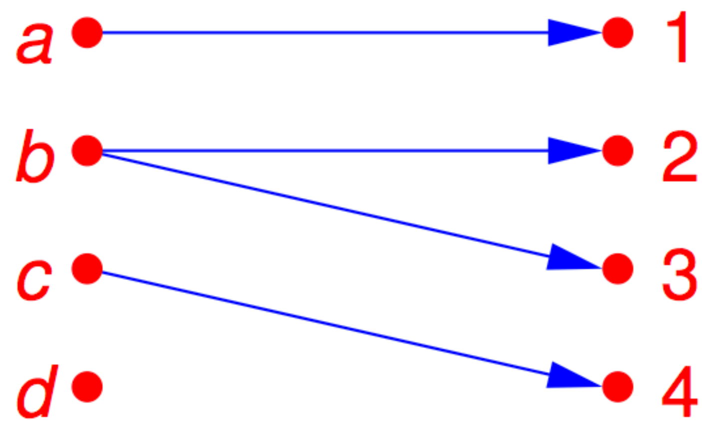
Example

Given

$$A = \{a, b, c, d\}, \qquad B = \{1, 2, 3, 4\},$$

which of the following are functions from A to B?





Example

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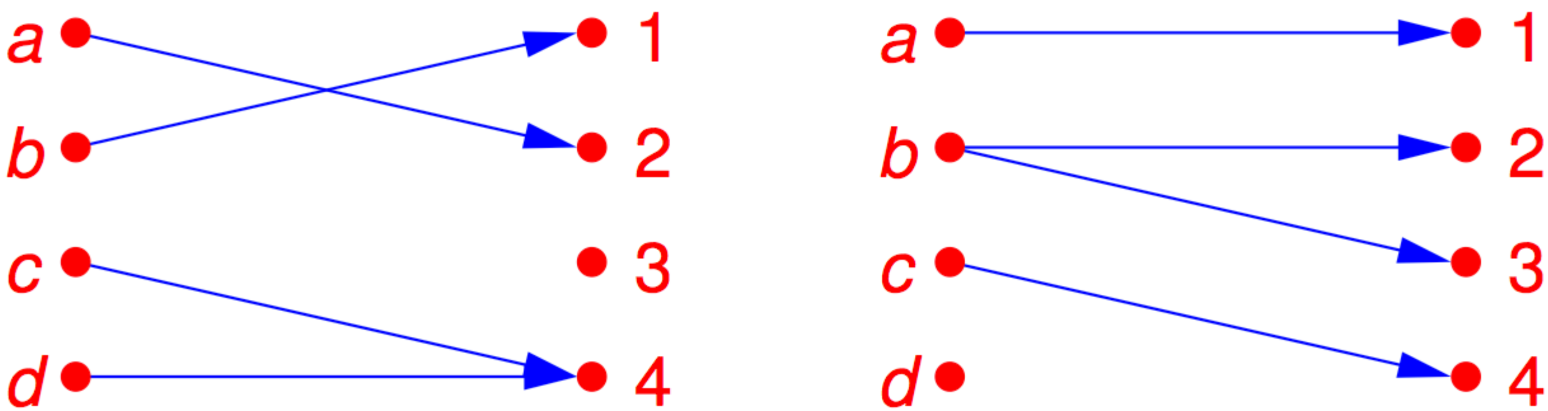
$$A = \{a, b, c, d\}, \qquad B = \{1, 2, 3, 4\},$$

which of the following are functions from A to B?



- The diagram on the left is a function. The codomain is a subset of B.
- The one on the right is not a function. The domain is not A. Moreover b is assigned (mapped) to two different elements of B.

Example

Which of the following are functions from $\mathbb{R}$ to $\mathbb{R}$?

- $f = \{(x, x) : x \in \mathbb{R}\}$.
- $g = \{(x, x^2) : x \in \mathbb{R}\}$.
- $h = \{(x^2, x) : x \in \mathbb{R}\}$.
- $j = \{(x, x^3) : x \in \mathbb{R}\}$.
- $k = \{(x^3, x) : x \in \mathbb{R}\}$.
- $m = \{(x, y) : x, y \in \mathbb{R}\}$.

Example

Which of the following are functions from $\mathbb{R}$ to $\mathbb{R}$?

- $f = \{(x, x) : x \in \mathbb{R}\}$. Function.
- $g = \{(x, x^2) : x \in \mathbb{R}\}$. Function.
- $h = \{(x^2, x) : x \in \mathbb{R}\}$. **Not** a function; $(4, -2)$ and $(4, 2)$ both in h.
- $j = \{(x, x^3) : x \in \mathbb{R}\}$. Function.
- $k = \{(x^3, x) : x \in \mathbb{R}\}$. Function.
- $m = \{(x, y) : x, y \in \mathbb{R}\}$. **Not** a function; $(1, 1)$ and $(1, 2)$ both in m.

Definitions

- A function is called **real-valued** if its codomain is the set of real numbers.
- A function is called **integer-valued** if its codomain is the set of integers.
- Two real-valued functions or two integer-valued functions with the same domain can be added or multiplied.

Example:

- Let f: R → R and g: R → R such that $f(x) = x^2$ and $g(x) = x - x^2$. What are the functions f+g and f.g?
- Solution:

  $(f+g)(x) = f(x) + g(x) = x^2 + (x - x^2) = x.$

  $f.g(x) = f(x).g(x) = x^2 . (x - x^{2)} = x^3 - x^4.$

Equality of functions

- Definition: Two functions $f: A \rightarrow B$ and $g: C \rightarrow D$ are equal if $A = C$ and $\forall x \in A$, $f(x) = g(x)$.

- Note that

  $f: Z \rightarrow N$ and $g: Z \rightarrow Z$, $f(x) = |x| + 2$, $g(x) = |x| + 2$

  are equal.

More Definitions (2)

- **Definition**: Let f: A$\longrightarrow$ B and S $\subseteq$A. The image of the set S is the subset of B that consists of all the images of the elements of S. We denote the image of S by f(S), so that

$$f(S)=\{ f(s) \mid \forall s \in S \}$$

- Note there that the image of S is a set and not an element.

Image of a set: Example

- Let:
  - $A = \{a_1,a_2,a_3,a_4,a_5\}$
  - $B = \{b_1,b_2,b_3,b_4,b_5\}$
  - $f=\{(a_1,b_2), (a_2,b_3), (a_3,b_3), (a_4,b_1), (a_5,b_4)\}$
  - $S=\{a_1,a_3\}$
  - $f(S) = \{b_2, b_3\}$
- Draw a diagram for f
- What is the:
  - Domain, co-domain, range of f?
  - Image of S, f(S)?

Section 12.2
(Injective and Surjective functions)

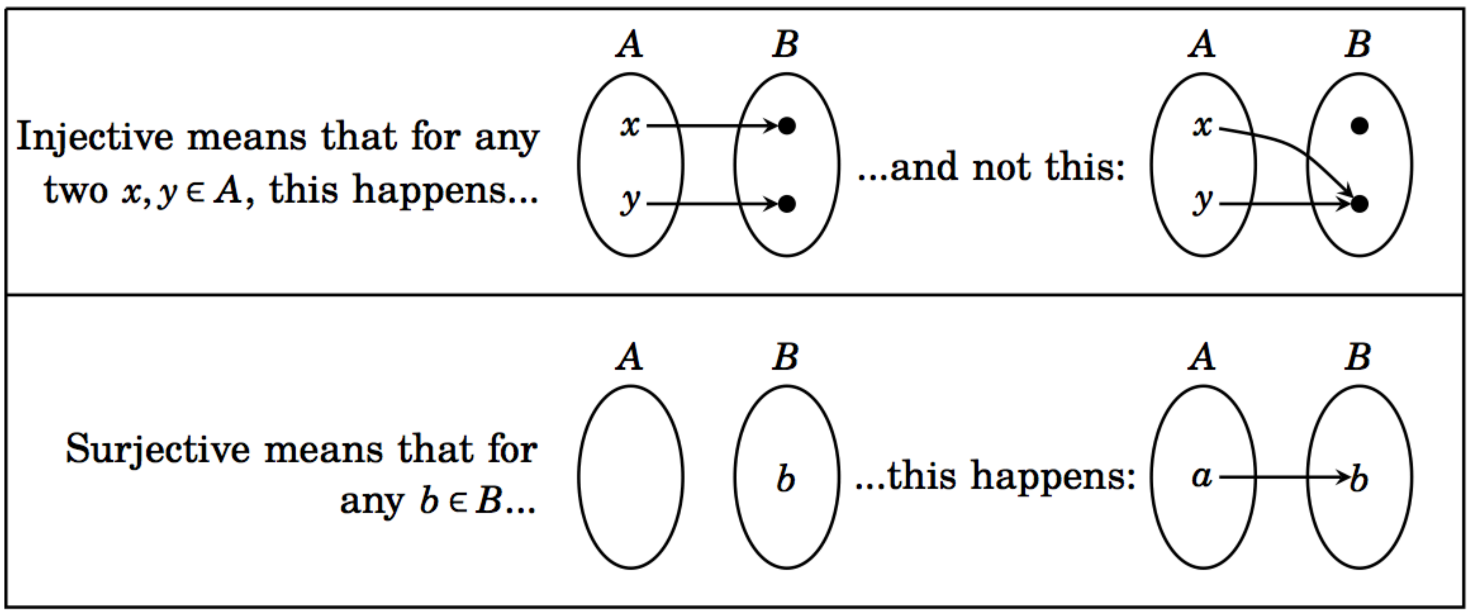
- In the literature
  - injective and one-to-one mean the same
  - surjective and onto mean the same
  - bijective and one-to-one correspondence mean the same.

Section 12.2
(Injective and Surjective functions)

- Definitions:

Definition 12.4 A function $f : A \to B$ is:

1. **injective** (or one-to-one) if for every $x, y \in A$, $x \neq y$ implies $f(x) \neq f(y)$;
2. **surjective** (or onto) if for every $b \in B$ there is an $a \in A$ with $f(a) = b$;
3. **bijective** if f is both injective and surjective.



Example

- Consider the following functions on the students in macm class. Under what conditions is the function one-to-one if it assigns to a student his or her
    1. cell phone number
    2. student id
    3. final grade in the class
    4. home town

Example

- Consider the following functions on the students in macm class. Under what conditions is the function one-to-one if it assigns to a student his or her
  1. cell phone number **one-to-one**
  2. student id **one-to-one**
  3. final grade in the class **generally not one-to-one unless each student gets a unique grade**
  4. home town **one-to-one if each student comes from a different town**

Injective functions
One-to-one functions

Which of the following are functions from $A = \{a, b, c\}$ to $B = \{1, 2, 3\}$? Of those, which are injective?

- $f = \{(a, 1), (b, 2), (c, 3)\}$.
- $g = \{(a, 1), (a, 2), (b, 3), (c, 3)\}$.
- $h = \{(a, 2), (b, 2), (c, 2)\}$.
- $j = \{(a, 3), (b, 1), (c, 2)\}$.

Which of the following are functions? Of those, which are injective?

- $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$.
- $g : \mathbb{N} \to \mathbb{Z}$, $g(x) = x^2$.
- $h : \mathbb{Z} \to \mathbb{N}$, $h(x) = x^2$.
- $j : \mathbb{N} \to \mathbb{N}$, $j(x) = x^2$.

Injective functions
One-to-one functions

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- $f = \{(a, 1), (b, 2), (c, 3)\}$. Function, injective.
- $g = \{(a, 1), (a, 2), (b, 3), (c, 3)\}$. **Not** a function.
- $h = \{(a, 2), (b, 2), (c, 2)\}$. Function, **not** injective.
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Which of the following are functions? Of those, which are injective?

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- $h : \mathbb{Z} \to \mathbb{N}$, $h(x) = x^2$.
- $j : \mathbb{N} \to \mathbb{N}$, $j(x) = x^2$.

How to show a function f: A ⟶ B is injective?

Direct approach:

Suppose $x, y \in A$ and $x \neq y$.

$\vdots$

Therefore $f(x) \neq f(y)$.

Contrapositive approach:

Suppose $x, y \in A$ and $f(x) = f(y)$.

$\vdots$

Therefore $x = y$.

$$\forall\, x,y \in A\ [\{(x \neq y) \Longrightarrow (f(x) \neq f(y))\} \equiv \{(f(x) = f(y)) \Longrightarrow (x=y)\}]$$

Injective functions
One-to-one functions

Which of the following are functions from $A = \{a, b, c\}$ to $B = \{1, 2, 3\}$? Of those, which are injective?

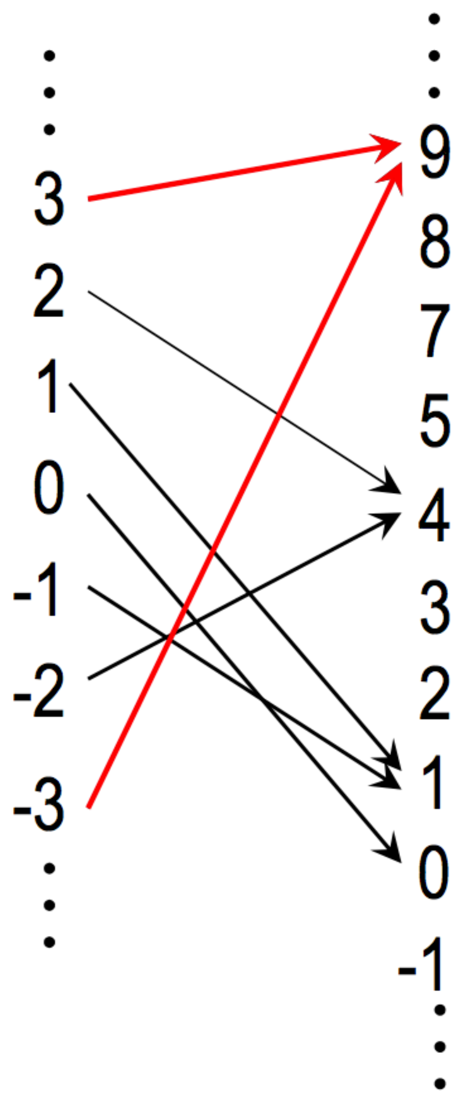
- $f = \{(a, 1), (b, 2), (c, 3)\}$. Function, injective.
- $g = \{(a, 1), (a, 2), (b, 3), (c, 3)\}$. **Not** a function.
- $h = \{(a, 2), (b, 2), (c, 2)\}$. Function, **not** injective.
- $j = \{(a, 3), (b, 1), (c, 2)\}$. Function, injective.

Which of the following are functions? Of those, which are injective?

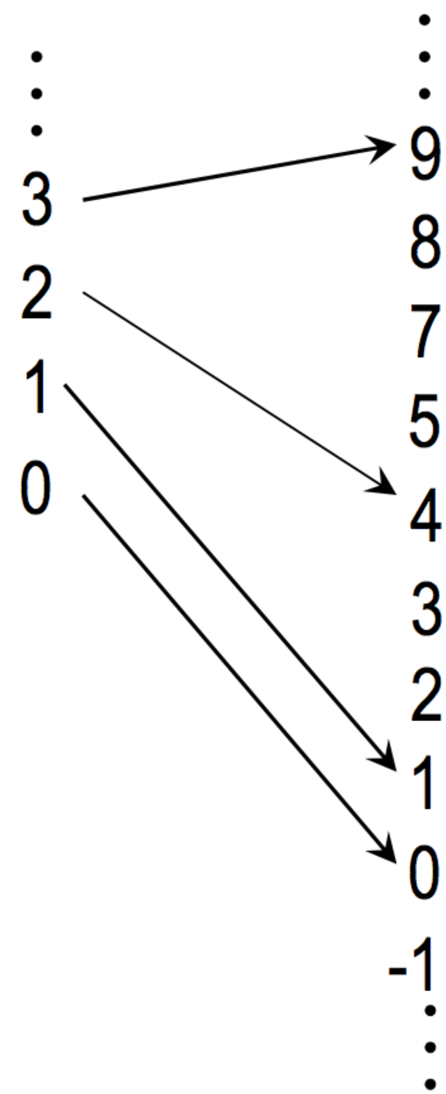
- $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$.
- $g : \mathbb{N} \to \mathbb{Z}$, $g(x) = x^2$.
- $h : \mathbb{Z} \to \mathbb{N}$, $h(x) = x^2$.
- $j : \mathbb{N} \to \mathbb{N}$, $j(x) = x^2$.

- Let's consider the function $f(x) = x^2$ on $\mathbb{Z}$

  Is it injective?



No!



Yes! on $\mathbb{N}$

Injective functions
One-to-one functions

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- $j = \{(a, 3), (b, 1), (c, 2)\}$. Function, injective.

Which of the following are functions? Of those, which are injective?

- $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$. Function, **not** injective; $f(-2) = f(2)$.
- $g : \mathbb{N} \to \mathbb{Z}$, $g(x) = x^2$. Function, injective.
- $h : \mathbb{Z} \to \mathbb{N}$, $h(x) = x^2$. **Not** a function; $f(0)$ is not defined.
- $j : \mathbb{N} \to \mathbb{N}$, $j(x) = x^2$. Function, injective.

- Recall:

Definition 12.4 A function $f: A \to B$ is:

1. **injective** (or one-to-one) if for every $x, y \in A$, $x \neq y$ implies $f(x) \neq f(y)$;
2. **surjective** (or onto) if for every $b \in B$ there is an $a \in A$ with $f(a) = b$;
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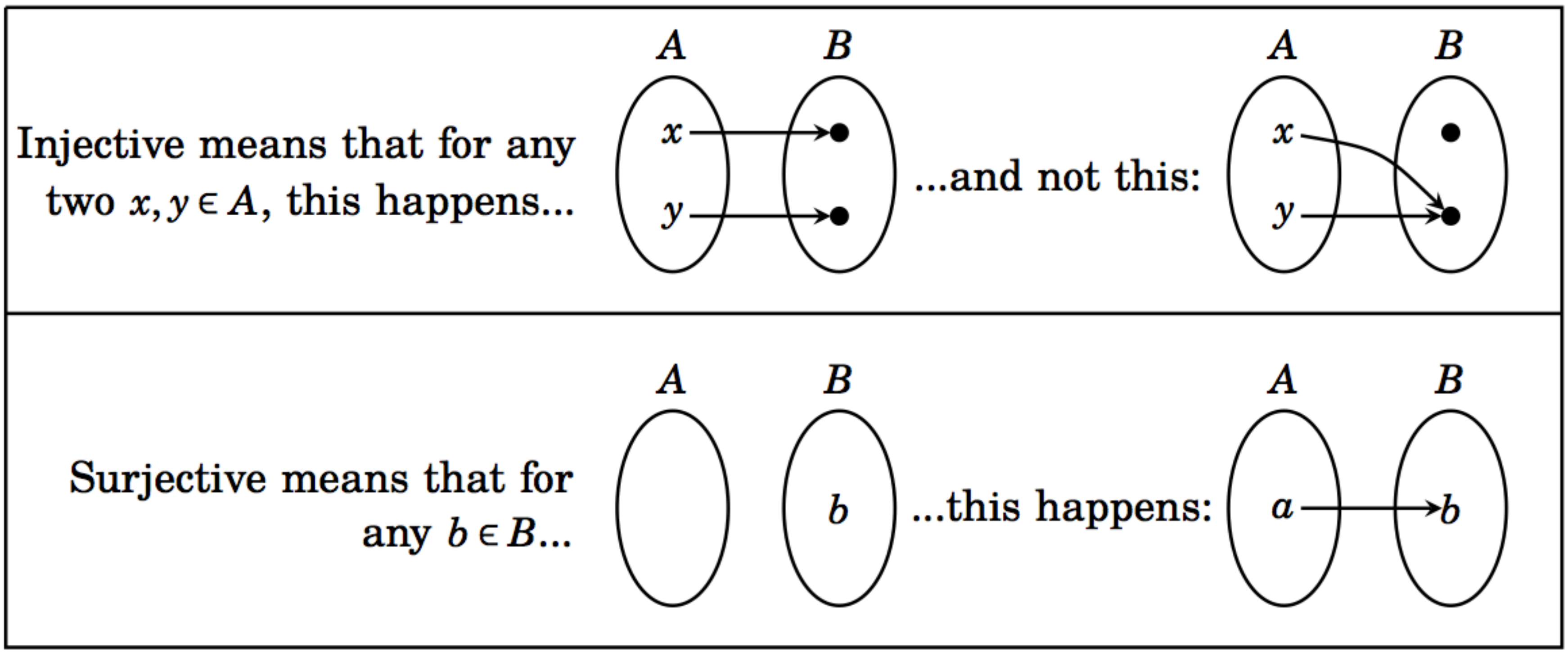
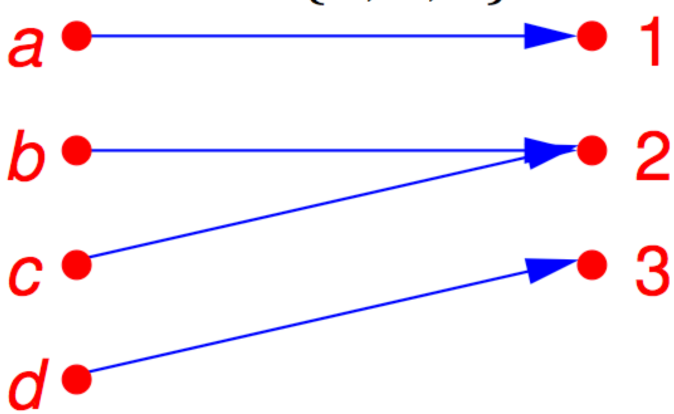


Surjective functions
Onto functions
range(f) = codomain(f)

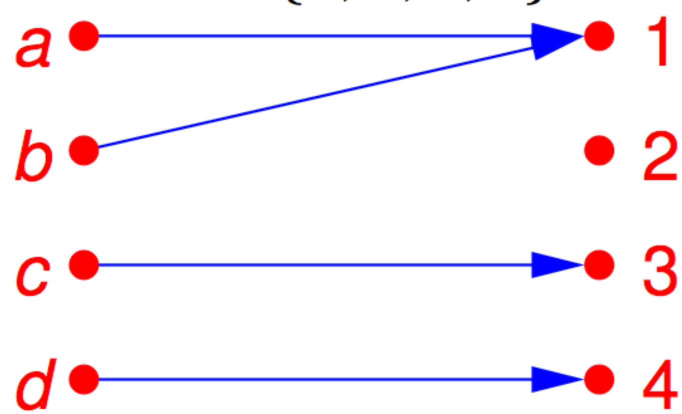
Suppose $A = \{a, b, c, d\}$.

$B = \{1, 2, 3\}$



surjective

$B = \{1, 2, 3, 4\}$



not surjective

range(f) ≠ codomain(f)

Surjective functions
Onto functions

Which of the following functions from $A = \{a, b, c\}$ to $B = \{1, 2, 3\}$ are surjective?

- $f = \{(a, 1), (b, 2), (c, 3)\}$.
- $h = \{(a, 1), (b, 2), (c, 2)\}$.
- $j = \{(a, 3), (b, 1), (c, 2)\}$.

Which of the following functions are surjective?

- $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$.
- $g : \mathbb{N} \to \mathbb{Z}$, $g(x) = x^2$.
- $j : \mathbb{N} \to \mathbb{N}$, $j(x) = x^2$.

Surjective functions
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- $f = \{(a, 1), (b, 2), (c, 3)\}$. Surjective.
- $h = \{(a, 1), (b, 2), (c, 2)\}$. **Not** surjective.
- $j = \{(a, 3), (b, 1), (c, 2)\}$. Surjective.

Which of the following functions are surjective?

- $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$.
- $g : \mathbb{N} \to \mathbb{Z}$, $g(x) = x^2$.
- $j : \mathbb{N} \to \mathbb{N}$, $j(x) = x^2$.

How to show a function f: A⟶ B is surjective?

Suppose $b \in B$.
[Prove there exists $a \in A$ for which $f(a) = b$.]

$\forall\, b \in B\ [\exists a \in A\ (f(a) = b)]$

Surjective functions
Onto functions

Which of the following functions from $A = \{a, b, c\}$ to $B = \{1, 2, 3\}$ are surjective?

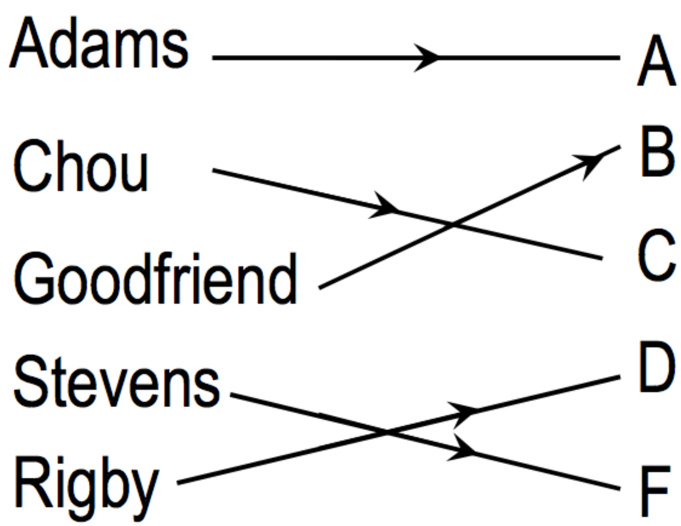
- $f = \{(a, 1), (b, 2), (c, 3)\}$. Surjective.
- $h = \{(a, 1), (b, 2), (c, 2)\}$. **Not** surjective.
- $j = \{(a, 3), (b, 1), (c, 2)\}$. Surjective.

Which of the following functions are surjective?

- $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$. **Not** surjective; no x for which $f(x) = -2$.
- $g : \mathbb{N} \to \mathbb{Z}$, $g(x) = x^2$. **Not** surjective; no x for which $f(x) = -2$.
- $j : \mathbb{N} \to \mathbb{N}$, $j(x) = x^2$. **Not** surjective (but injective)

Bijections

- A function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto.

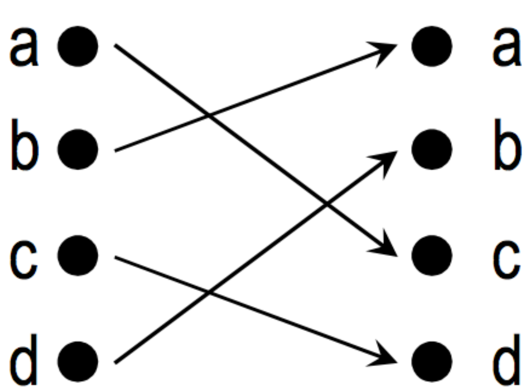


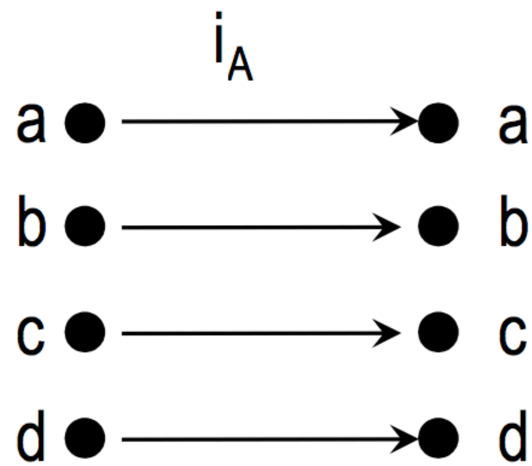
- If there is a bijection from a set A to a set B, then these sets in a certain sense are equal or identical.

Bijections (cntd)

- Numerical functions:
  - $f(x) = x + 1$ is a bijection on $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, but not on $\mathbb{N}$
  - $f(x) = x^2$ is a bijection on $\mathbb{R}^+_{;}$ but is not on any other numerical set

- A bijection from a set A to the same set A is called a permutation of A

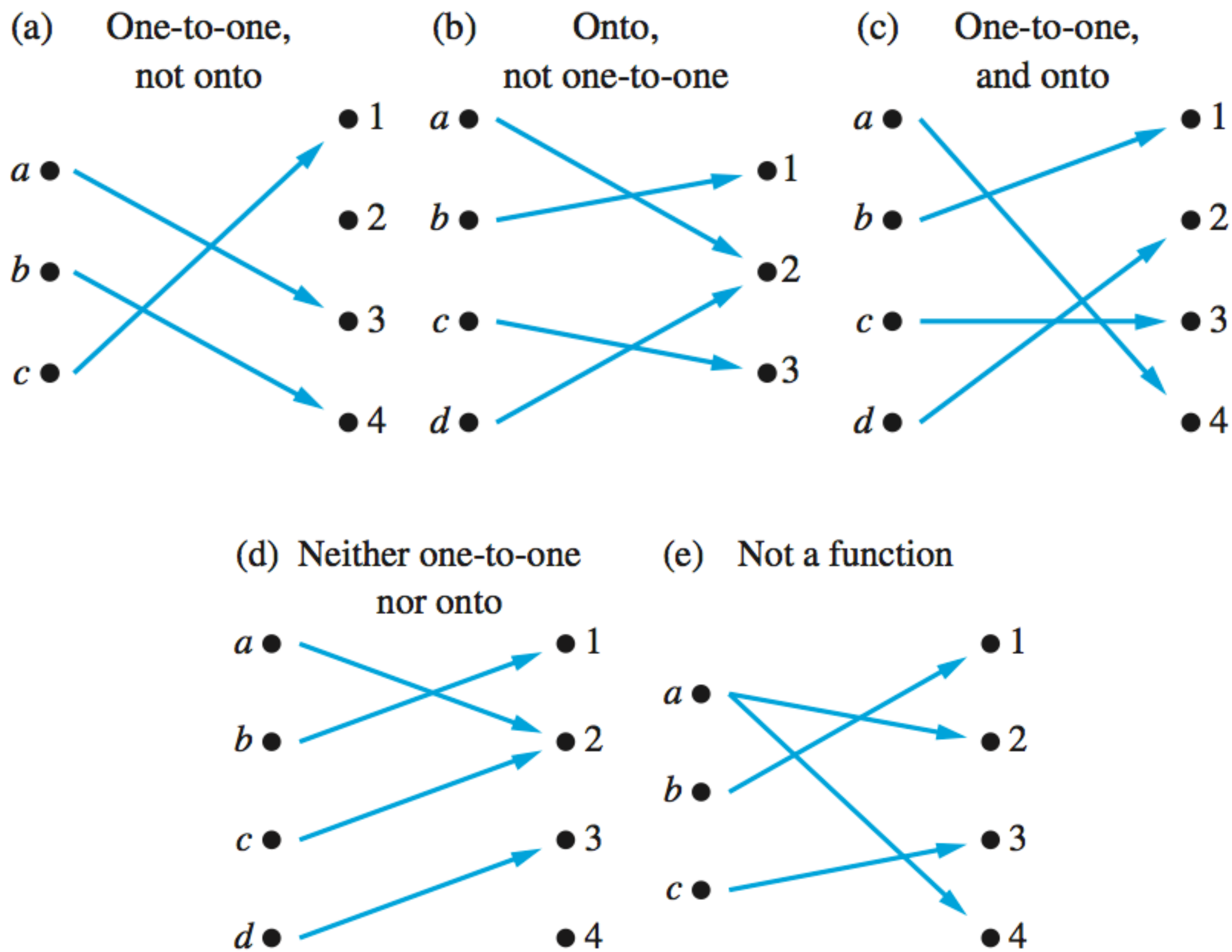




- The identity function on a set A is the function $i_A : A \to A$, where

$$i_A(x) = x$$

Examples of different types of correspondences



Exercise 1

- Let $f: Z \rightarrow Z$ be defined by

$$f(x)=2x-3$$

- What is the domain, codomain, range of f?
- Is f one-to-one (injective)?
- Is f onto (surjective)?
- Clearly, domain(f)=Z.
- To see what the range is, note that:

$$\begin{aligned} b \in \text{range}(f) &\Rightarrow b=2a-3, \text{ for some } a \in Z \\ &\Rightarrow b=2(a-2)+1 \\ &\Rightarrow b \text{ is odd} \end{aligned}$$

Exercise 1 (cont'd)

- Thus, the range is the set of all odd integers
- Since the range and the codomain are different (i.e., $\text{range}(f) \neq Z$), we can conclude that **f** is not onto (surjective)
- However, f is one-to-one injective. (Contrapositive approach)
  - For $x, y \in Z$, we can write

$$f(x) = f(y) \Rightarrow 2x - 3 = 2y - 3 \Rightarrow x = y. \quad \text{QED}$$

Exercise 2

- Let f be as before

$$f(x)=2x-3$$

  but now we define $f:N \rightarrow N$
- What is the domain and range of f?
- Is f onto (surjective)?
- Is f one-to-one (injective)?

- By changing the domain and codomain of *f*, *f* is not even a function anymore. Indeed, $f(1)=2\cdot 1-3=-1 \notin N$

Exercise 3

- Let $f: Z \rightarrow Z$ be defined by

$$f(x) = x^2 - 5x + 5$$

- Is this function
  - One-to-one?
  - Onto?

Exercise 3: Answer

- It is not one-to-one (injective)

$f(x_1)=f(x_2) \Rightarrow x_1^2-5x_1+5=x_2^2 - 5x_2+5 \Rightarrow x_1^2 - 5x_1 = x_2^2 - 5x_2$

$\Rightarrow x_1^2 - x_2^2 = 5x_1 - 5x_2 \Rightarrow (x_1 - x_2)(x_1 + x_2) = 5(x_1 - x_2)$

$\Rightarrow (x_1 + x_2) = 5$

Many $x_1,x_2 \in Z$ satisfy this equality. There are thus an infinite number of solutions. In particular, f(2)=f(3)=-1

- It is also not onto (surjective).

The function is a parabola with a global minimum at (5/2,-5/4). Therefore, the function fails to map to any integer less than -1

- What would happen if we changed the domain/ codomain?

Exercise 3: Answer

- It is not one-to-one (injective)

$f(x_1)=f(x_2) \Rightarrow x_1^2-5x_1+5=x_2^2 - 5x_2+5 \Rightarrow x_1^2 - 5x_1 = x_2^2 - 5x_2$

$\Rightarrow x_1^2 - x_2^2 = 5x_1 - 5x_2 \Rightarrow (x_1 - x_2)(x_1 + x_2) = 5(x_1 - x_2)$

$\Rightarrow (x_1 + x_2) = 5$

Many $x_1,x_2 \in Z$ satisfy this equality. There are thus an infinite number of solutions. In particular, f(2)=f(3)=-1

- It is also not onto (surjective).

The function is a parabola with a global minimum at (5/2,-5/4). Therefore, the function fails to map to any integer less than -1

- What would happen if we changed the domain/codomain?
- The function is one-to-one and onto when

domain(f) = $\{x \in R \mid x > 4\}$; codomain(f) = $\{y \in R \mid y > 0\}$

Exercise 4

- Let $f: Z \rightarrow Z$ be defined by

$$f(x) = 2x^2 + 7x$$

- Is this function
  - One-to-one (injective)?
  - Onto (surjective)?
- Again, this is a parabola, it cannot be onto.

Exercise 4: Answer

- f(x) is one-to-one! Indeed:

  $f(x_1)=f(x_2) \Rightarrow 2x_1^2+7x_1=2x_2^2 + 7x_2 \Rightarrow 2x_1^2 - 2x_2^2 = 7x_2 - 7x_1$

  $\Rightarrow 2(x_1 - x_2)(x_1 + x_2) = 7(x_2 - x_1) \Rightarrow 2(x_1 + x_2) = -7 \Rightarrow$

  $(x_1 + x_2) = -7$

  $\Rightarrow (x_1 + x_2) = -7/2$

  But $-7/2 \notin Z$. Therefore it must be the case that $x_1 = x_2$.

  It follows that f is a one-to-one function. QED

- f(x) is not surjective because f(x)=1 does not exist

  $2x^2 +7x=1 \Rightarrow x(2x +7)=1$ the product of two integers is 1 if both integers are 1 or -1

  $x=1 \Rightarrow (2x+7)=1 \Rightarrow 9 =1$, impossible

  $x=-1 \Rightarrow -1(-2+7)=1 \Rightarrow -5=1$, impossible

Exercise 5

- Let $f: Z \rightarrow Z$ be defined by

$$f(x) = 3x^3 - x$$

- Is this function
  - One-to-one (injective)?
  - Onto (surjective)?

Exercise 5: f is one-to-one

- To check if f is one-to-one, again we suppose that for $x_1,x_2 \in Z$ we have $f(x_1)=f(x_2)$

  $f(x_1)=f(x_2) \Rightarrow 3x_1^3-x_1=3x_2^3-x_2$

  $\Rightarrow 3x_1^3 - 3x_2^3 = x_1 - x_2$

  $\Rightarrow 3\ (x_1 - x_2)(x_1^2+x_1x_2+x_2^2)= (x_1 - x_2)$

  $\Rightarrow (x_1^2+x_1x_2+x_2^2)= 1/3$

  which is impossible because $x_1,x_2 \in Z$

  thus, f is one-to-one

Exercise 5: f is <u>not</u> onto

- Consider the counterexample f(a)=1
- If this were true, we would have

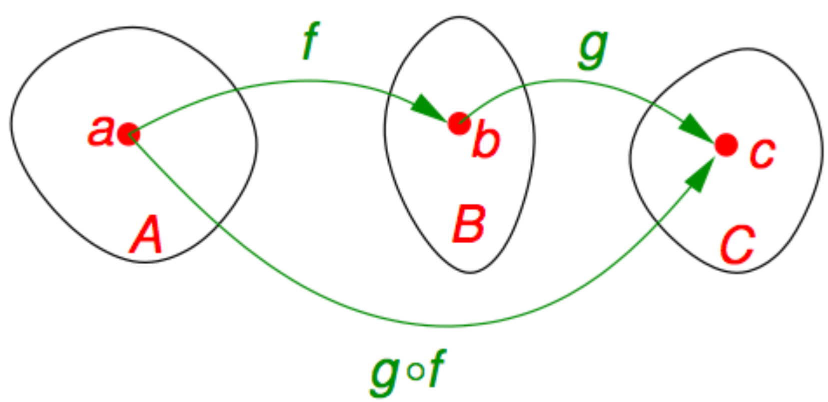
  $3a^3 - a = 1 \Rightarrow a(3a^2 - 1) = 1$ where a and $(3a^2 - 1) \in Z$
- The only time we can have the product of two integers equal to 1 is when they are both equal to 1 or -1
- Neither 1 nor -1 satisfy the above equality
  - Thus, we have identified $1 \in Z$ that does not have an antecedent and f is not onto (surjective)

Practice Problems

- Section 12.1: 1, 2, 3, 6, 7, 11
- Section 12.2: 1, 2, 4, 5, 9, 10, 13, 15, 17

Composition (Section 12.4)

- Definition:
  - Suppose $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions with the property that the **codomain(f)** equals the **domain(g)**.
  - The **composition of f and g** is another function, denoted as $g \circ f$ and defined as follows: If $x \in A$, then $g \circ f(x) = g(f(x))$. This function $g(f(x))$ sends (maps) elements of A to elements of C, so $g \circ f: A \rightarrow C$.



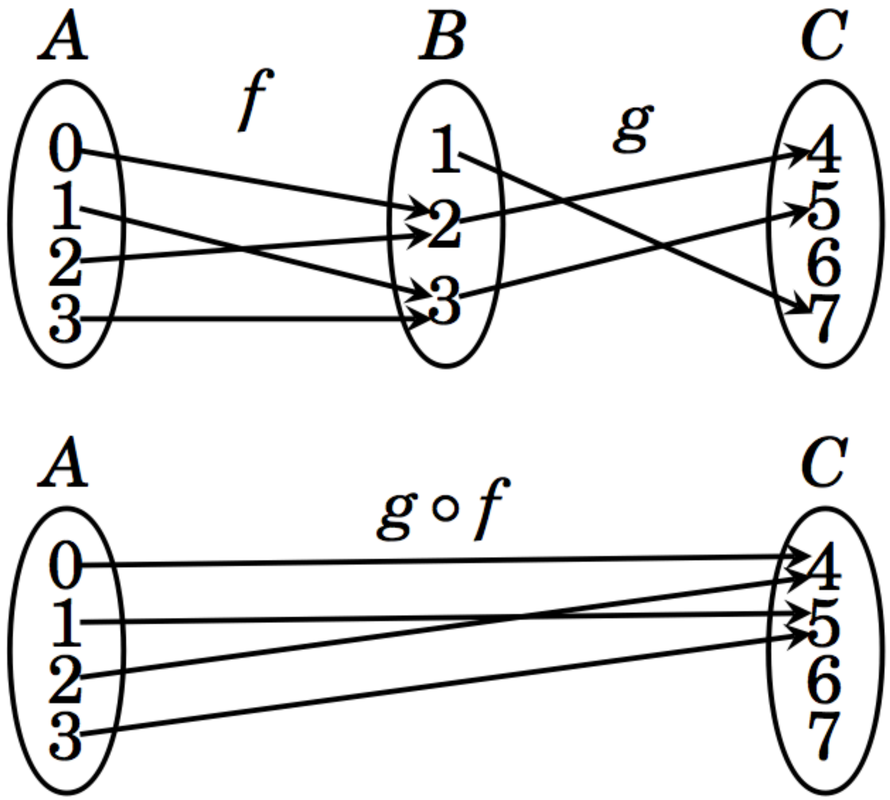


Figure 12.5. Composition of two functions

Composition: Example 1

- Let f, g be two functions on $R \rightarrow R$ defined by

$$f(x) = 2x - 3$$

$$g(x) = x^2 + 1$$

- What are $f \circ g$ and $g \circ f$?

Composition: Example 1 (cont'd)

- Given $f(x) = 2x - 3$ and $g(x) = x^2 + 1$
- $(f \circ g)(x) = f(g(x)) = f(x^2+1) = 2(x^2+1)-3$
  $= 2x^2 - 1$
- $(g \circ f)(x) = g(f(x)) = g(2x-3) = (2x-3)^2 +1$
  $= 4x^2 - 12x + 10$

Thus the composition of functions is not commutative.

Associativity

- **Lemma**: The composition of functions is an associative operation, that is

$$(f \circ g) \circ h = f \circ (g \circ h)$$

Composition: Injection and Surjection

- Theorem:

  Suppose $f: A \rightarrow B$ and $g: B \rightarrow C$. If both **f** and **g** are injective then $g \circ f$ is injective. If both **f** and **g** are surjective, $g \circ f$ is also surjective.

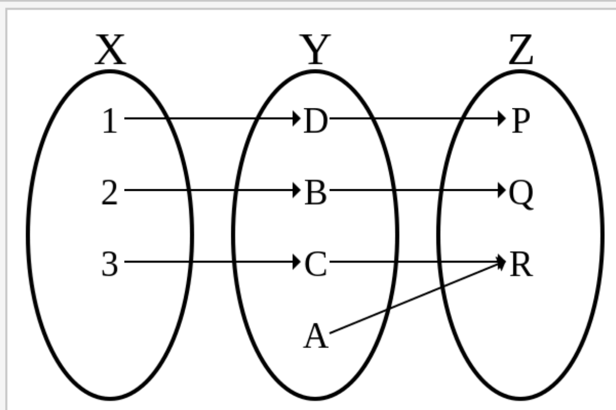
Proof of:

Suppose $f: A \rightarrow B$ and $g: B \rightarrow C$. If both **f** and **g** are injective then $g \circ f$ is injective.

- Here both f and g are injective.
- To see that $g \circ f$ is injective, we must show that $g(f(x)) = g(f(y))$ implies $x = y$.
- Suppose $g(f(x)) = g(f(y))$.
- Since g is injective, $f(x) = f(y)$.
- Since $f(x) = f(y)$, and f is injective, $x = y$.
- Hence $g(f(x)) = g(f(y))$ implies $x = y$.
- Therefore, $g \circ f$ is injective.

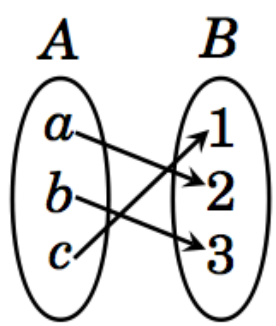
Conversely,

- Suppose f : X → Y and g: Y → Z.
- If the composition $g \circ f$ of two functions is bijective, we can only say that f is injective and g is surjective.

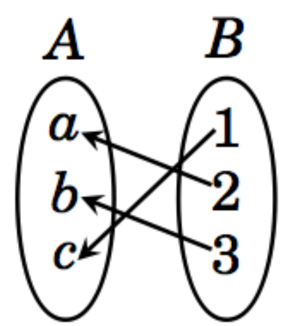


A bijection composed of an injection (left) and a surjection (right).

- Definition: Given a relation R from A to B, the **inverse relation** of R is the relation from B to A defined as $R^{-1} = \{(y,x): (x,y) \in R\}$.



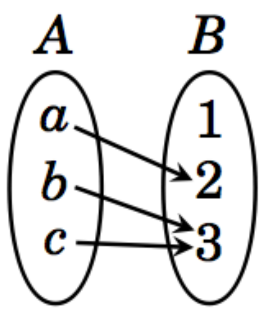
$f = \{(a,2),(b,3),(c,1)\}$



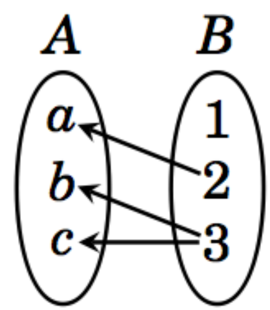
$f^{-1} = \{(2,a),(3,b),(1,c)\}$

- f is function, but f^{-1} is also a function.

- Definition: Given a relation R from A to B, the **inverse relation** of R is the relation from B to A defined as $R^{-1} = \{(y,x): (x,y) \in R\}$.



$g = \{(a,2),(b,3),(c,3)\}$

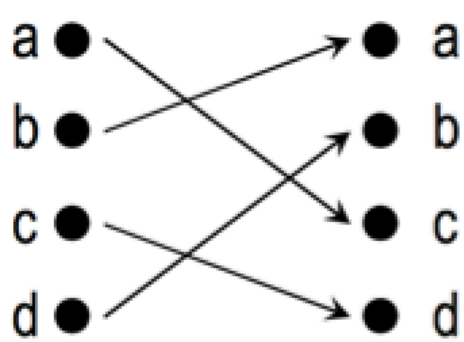


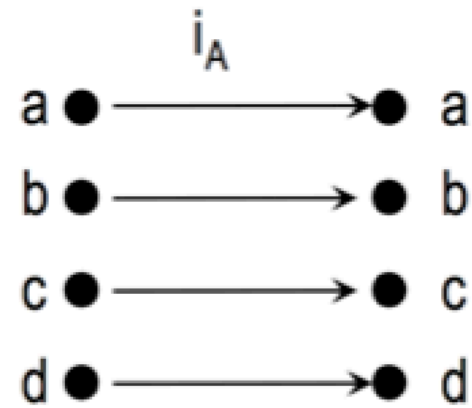
$g^{-1} = \{(2,a),(3,b),(3,c)\}$

- g is function, but g^{-1} is not a function.

Inverse Functions

- A bijection from a set A to the same set A is called a permutation of A





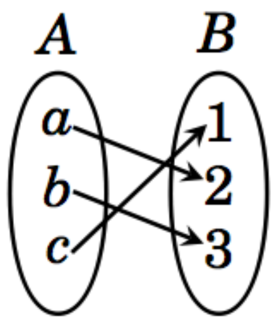
- The identity function on a set A is the function $i_A : A \to A$, where

$$i_A(x) = x$$

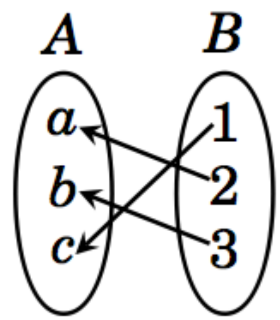
- **Definition**: Suppose $f : A \rightarrow B$ is bijective. The **inverse function** of f is the th function that assigns to each element $b \in B$, an element (preimage) $a \in A$ such that $f(a) = b$. We denote the inverse function $f^{-1}: B \rightarrow A$ and can write

$$f^{-1}(b) = a \text{ when } f(a) = b.$$

- $f^{-1} \circ f = i_A$
- $f \circ f^{-1} = i_B$



$f = \{(a,2),(b,3),(c,1)\}$



$f^{-1} = \{(2,a),(3,b),(1,c)\}$

- $f^{-1} \circ f = \{(a,a), (b,b), (c,c)\}$
- $f \circ f^{-1} = \{(1,1), (2,2), (3,3)\}$

Example

- Find the inverse of the function $f : R \rightarrow R$ defined as $f(x) = x^3 + 1$, if it exists.
- We first show that f is bijective.
  - f is one-to-one:
  - $f(x) = f(y)$ implies $(x^3 + 1) = (y^3 + 1)$
  - i.e $(x-y)(x^2 + xy + y^2) = 0$.
  - Since $(x^2 + xy + y^2)$ is not zero for any $x \neq y$, therefore $f(x) = f(y)$ implies $x = y$.
  - Therefore, f is one-to-one.

Example

- Find the inverse of the function $f : R \rightarrow R$ defined as $f(x) = x^3 + 1$, if it exists.
- We first show that f is bijective.
  - f is one-to-one:
  - We now show that f is onto.
  - For any y of R, we need to find x such that $f(x) = y$.
  - i.e. find x such that $x^3 + 1 = y$, i.e $x = (y-1)^{1/3}$. Thus we see that $f((y-1)^{1/3}) = y$ for any y.
  - Therefore, f is onto.
  - Thus f is bijective

Example

- Find the inverse of the function $f : R \rightarrow R$ defined as $f(x) = x^3 + 1$, if it exists.
- We now find $f^{-1}(x)$.
  - We are interested in computing y such that $f^{-1}(x) = y$.
  - i.e. computing y such that $f(y) = x$.
  - i.e. computing y such that $y^3 + 1 = x$.
  - i.e. $y = (x - 1)^{1/3}$.
  - Thus $f^{-1}(x) = (x - 1)^{1/3}$.
  - Note that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$.

Another Example

- Consider the function g: Z x Z $\rightarrow$ X x Z defined by g(m,n) = (m+n, m + 2n). g is bijective (show that). Find its inverse.
- We want to find (x,y) $\in$ Z x Z such that g^{-1}(m,n) = (x,y).
  - i.e. (m,n) = g(x,y)
  - i.e. (m,n) = (x +y, x + 2y)
  - i.e. m = x+y and n = x+ 2y
  - i.e. x = 2m – n and y = n – m.
  - Thus g^{-1}(m,n) = (2m –n, n-m).

Important Functions: Absolute Value

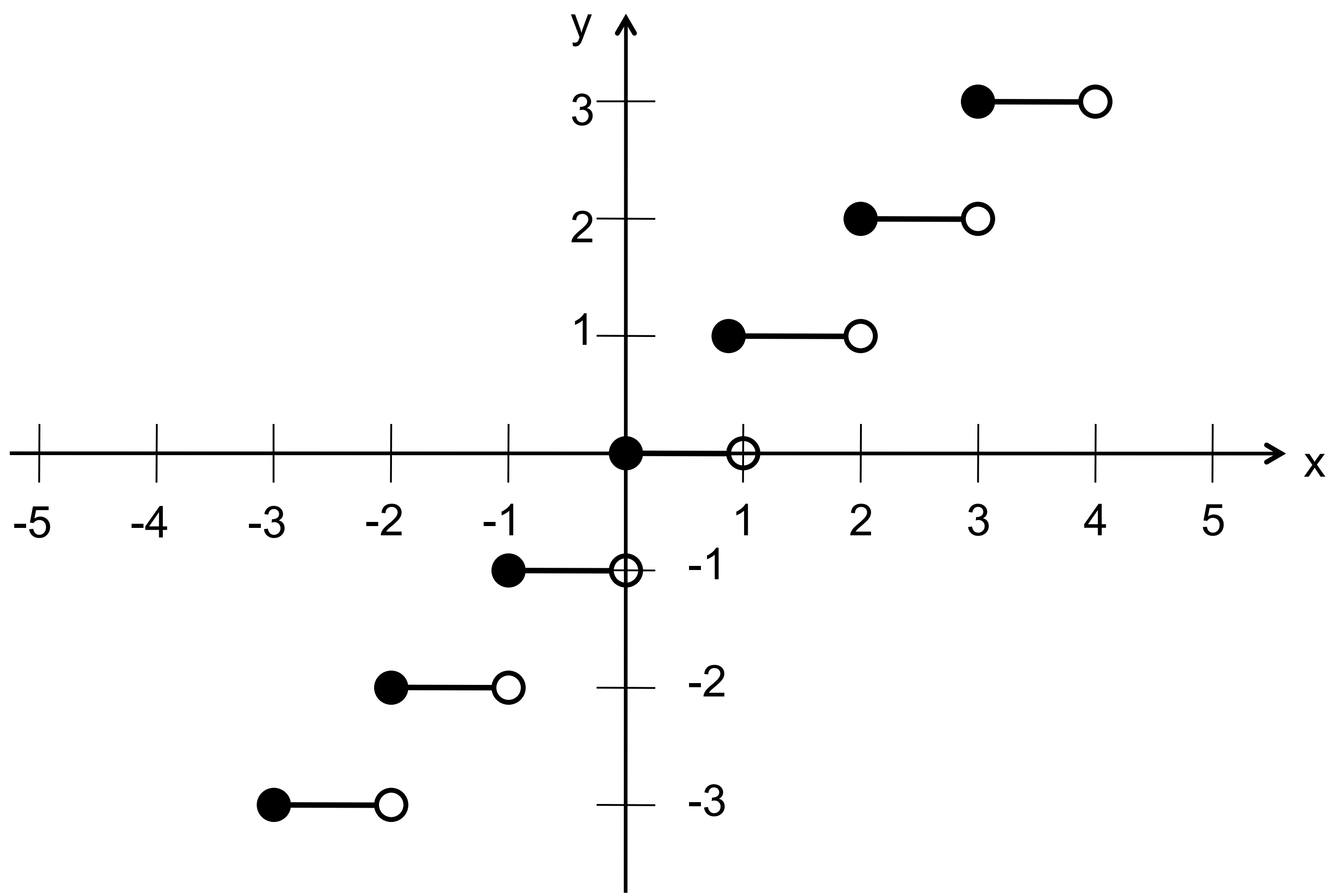
- **Definition**: The absolute value function, denoted $|x|$, $f : R \rightarrow \{y \in R \mid y \geq 0\}$. Its value is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

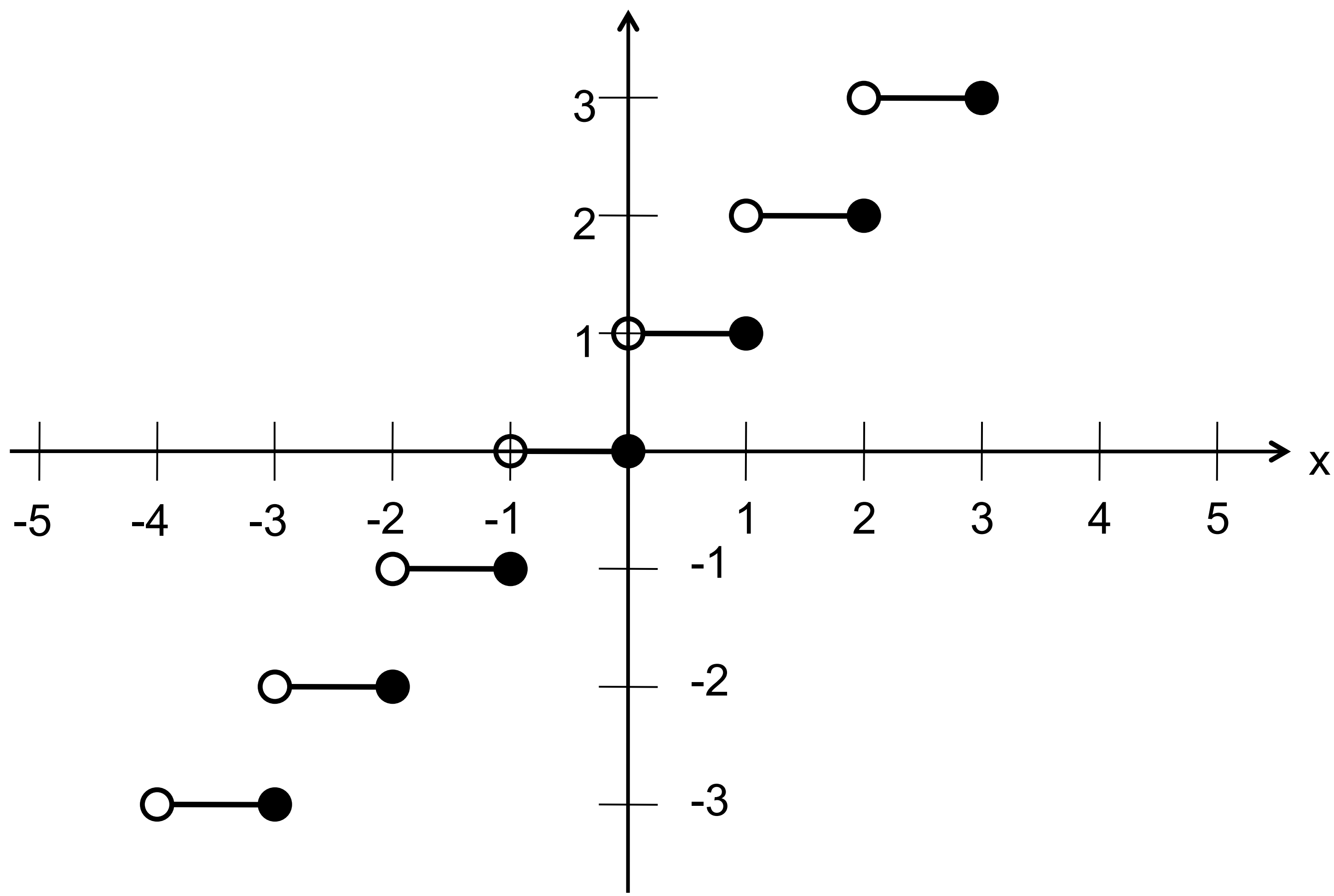
Important Functions: Floor & Ceiling

- **Definitions**:
  - The <u>floor function</u>, denoted $\lfloor x \rfloor$, is a function $R \rightarrow Z$. Its values is the <u>largest integer</u> that is less than or equal to x
  - The ceiling function, denoted $\lceil x \rceil$, is a function $R \rightarrow Z$. Its values is the <u>smallest integer</u> that is greater than or equal to x .

Important Functions: Floor



Important Functions: Ceiling



Important Function: Factorial

- The factorial function gives us the number of permutations (that is, uniquely ordered arrangements) of a collection of n objects
- **Definition**: The factorial function, denoted n!, is a function $N \rightarrow N^+$. Its value is the product of the n positive integers

$$n! = \prod_{i=1}^{i=n} i = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$$

Summary

- Definitions & terminology
  - function, domain, co-domain, image, preimage , range, image of a set
- Properties
  - One-to-one (injective), onto (surjective), one-to-one correspondence (bijective)
- Operators
  - Composition
- Inverse functions
- Important functions
  - identity, absolute value, floor, ceiling, factorial

Practice Problems

- Section 12.4: 2, 5, 6, 7, 8, 10
- Section 12.5: 1, 2, 3, 6, 9
- Section 12.6: 2, 3, 4, 5