Counting and Proofs

Tutorial 3

February 25, 2015

***How many ways can the 11 distinct horses be lined up in a row?

***How many ways can the 11 distinct horses be lined up in a row?
Solution : 11!

***How many ways can the 11 distinct horses be lined up in a row?
Solution : 11!
**How many ways can the 11 distinct horses be lined up in a cycle?

***How many ways can the 11 distinct horses be lined up in a row?
Solution : 11!

**How many ways can the 11 distinct horses be lined up in a cycle?
Solution : Note that (in case of 3 horses) $< a, b, c >$ cyclic order is the same as $< b, c, a >$ and $< c, a, b >$. In this case, there is no position 1, position 2 or position 3 in the order. We need to scale 11! to accommodate this. The correct answer is 11!/11 which is 10!

***How many ways can the 11 distinct horses be lined up in a row?
Solution : 11!
**How many ways can the 11 distinct horses be lined up in a cycle?
Solution : Note that (in case of 3 horses) $< a, b, c >$ cyclic order is the same as $< b, c, a >$ and $< c, a, b >$. In this case, there is no position 1, position 2 or position 3 in the order. We need to scale 11! to accommodate this. The correct answer is 11!/11 which is 10!
***How many ways can the 11 identical horses on a cycle be painted so that three are brown, three are white and five are black?

***How many ways can the 11 distinct horses be lined up in a row?
Solution : 11!
**How many ways can the 11 distinct horses be lined up in a cycle?
Solution : Note that (in case of 3 horses) $< a, b, c >$ cyclic order is the same as $< b, c, a >$ and $< c, a, b >$. In this case, there is no position 1, position 2 or position 3 in the order. We need to scale 11! to accommodate this. The correct answer is 11!/11 which is 10!
***How many ways can the 11 identical horses on a cycle be painted so that three are brown, three are white and five are black?
Solution:

***How many ways can the 11 distinct horses be lined up in a row?
Solution : 11!
**How many ways can the 11 distinct horses be lined up in a cycle?
Solution : Note that (in case of 3 horses) $< a, b, c >$ cyclic order is the same as $< b, c, a >$ and $< c, a, b >$. In this case, there is no position 1, position 2 or position 3 in the order. We need to scale 11! to accommodate this. The correct answer is 11!/11 which is 10!
***How many ways can the 11 identical horses on a cycle be painted so that three are brown, three are white and five are black?
Solution: We have 10! ways to place 11 horses in a circle. For the 3 brown horses, we do not distinguish their order. Similar situation to the 3 white horses and 5 black horses. Thus the solution is $10!/(3! * 3! * 5!)$

***Make up a word problem in English whose answer is

***Make up a word problem in English whose answer is
Q(a) $C(25,7) * C(10,3)$

***Make up a word problem in English whose answer is
Q(a) $C(25,7) * C(10,3)$
Solution : How many ways can we built a team of 7 boys and 3 girls out of a class of 25 boys and 10 girls

***Make up a word problem in English whose answer is
Q(a) $C(25, 7) * C(10, 3)$
Solution : How many ways can we built a team of 7 boys and 3 girls out of a class of 25 boys and 10 girls
Q(b) $2^n - 2$

***Make up a word problem in English whose answer is

Q(a) $C(25, 7) * C(10, 3)$

Solution : How many ways can we built a team of 7 boys and 3 girls out of a class of 25 boys and 10 girls

Q(b) $2^n - 2$

Solution : How many n bit binary strings that have at least one 1 and at least one 0

***Use binomial theorem to prove the following

***Use binomial theorem to prove the following
Q(a): $\sum_{i=0}^{10} \binom{10}{i} = 2^{10}$

***Use binomial theorem to prove the following
Q(a): $\sum_{i=0}^{10} \binom{10}{i} = 2^{10}$
Solution

***Use binomial theorem to prove the following

Q(a): $\sum_{i=0}^{10} \binom{10}{i} = 2^{10}$

Solution

$\sum_{i=0}^{10} \binom{10}{i} = \binom{10}{0} + \binom{10}{1} + ... + \binom{10}{10}$

Consider the following:

$2^{10} = (1+1)^{10} = \binom{10}{0} * 1^{10} + \binom{10}{1} * 1^{10} + ... + \binom{10}{10} * 1^{10} =$
$\binom{10}{0} + \binom{10}{1} + ... + \binom{10}{10}$

***Use binomial theorem to prove the following

Q(a): $\sum_{i=0}^{10} \binom{10}{i} = 2^{10}$

Solution

$\sum_{i=0}^{10} \binom{10}{i} = \binom{10}{0} + \binom{10}{1} + ... + \binom{10}{10}$

Consider the following:

$2^{10} = (1+1)^{10} = \binom{10}{0} * 1^{10} + \binom{10}{1} * 1^{10} + ... + \binom{10}{10} * 1^{10} =$
$\binom{10}{0} + \binom{10}{1} + ... + \binom{10}{10}$

Q(b): $\sum_{i=0}^{50} \binom{100}{2i} = \sum_{i=1}^{50} \binom{100}{2i-1}$

***Use binomial theorem to prove the following

Q(a): $\sum_{i=0}^{10} \binom{10}{i} = 2^{10}$

Solution

$\sum_{i=0}^{10} \binom{10}{i} = \binom{10}{0} + \binom{10}{1} + ... + \binom{10}{10}$

Consider the following:

$2^{10} = (1+1)^{10} = \binom{10}{0} * 1^{10} + \binom{10}{1} * 1^{10} + ... + \binom{10}{10} * 1^{10} =$
$\binom{10}{0} + \binom{10}{1} + ... + \binom{10}{10}$

Q(b): $\sum_{i=0}^{50} \binom{100}{2i} = \sum_{i=1}^{50} \binom{100}{2i-1}$

Solution

Consider the following

$0 = (1-1)^{100} = \binom{100}{0} - \binom{100}{1} + \binom{100}{2} - ... + \binom{100}{100}$

We move all the negative item to the left side of the equation we get $\sum_{i=0}^{50} \binom{100}{2i} = \sum_{i=1}^{50} \binom{100}{2i-1}$

Section 4

Use the method of direct proof to prove the following statements.

Section 4

Use the method of direct proof to prove the following statements.
Q 6: Suppose $x, y, z \in \mathcal{Z}$. If $a|b$ and $a|c$, then $a|(b+c)$

Section 4

Use the method of direct proof to prove the following statements.
Q 6: Suppose $x, y, z \in \mathcal{Z}$. If $a|b$ and $a|c$, then $a|(b+c)$
Proof:

Section 4

Use the method of direct proof to prove the following statements.
Q 6: Suppose $x, y, z \in \mathcal{Z}$. If $a|b$ and $a|c$, then $a|(b+c)$

Proof:

Soppose $a|b$ and $a|c$

By definition of divisibility, this means $b = ak_1$ for some integer k_1 and $c = ak_2$ for some integer k_2

Add both equation produces $b + c = ak_1 + ak_2$ which is $b + c = (k_1 + k_2)a$

By definition of divisibility, this means $a|(b+c)$

Section 4

Use the method of direct proof to prove the following statements.

Section 4

Use the method of direct proof to prove the following statements.
Q 20: Suppose $n \in \mathcal{N}$, then the numbers $n! + 2, n! + 3, ...n! + n$ are all composite (i.e. has at least two factors other than 1 and the number itself). (Thus for any $n \geq 2$, one can find n consecutive composite numbers. This means there are arbitrarily large "gaps" between prime numbers)

Section 4

Use the method of direct proof to prove the following statements.
Q 20: Suppose $n \in \mathcal{N}$, then the numbers $n! + 2, n! + 3, ... n! + n$ are all composite (i.e. has at least two factors other than 1 and the number itself). (Thus for any $n \geq 2$, one can find n consecutive composite numbers. This means there are arbitrarily large "gaps" between prime numbers)

Proof:

Section 4

Use the method of direct proof to prove the following statements.
Q 20: Suppose $n \in \mathcal{N}$, then the numbers $n! + 2, n! + 3, \ldots n! + n$ are all composite (i.e. has at least two factors other than 1 and the number itself). (Thus for any $n \geq 2$, one can find n consecutive composite numbers. This means there are arbitrarily large "gaps" between prime numbers)

Proof:

Suppose $n \geq 2$.
For any other value of $n \geq 2$, 2 will always be a factor of $n!$, since $n! = n * (n-1) * \ldots * 2 * 1$.
$n, n-1, \ldots, 3$ are also factors of $n!$.
Any number $n! + k$ is a composite number since
$n! + k = k[(n \times (n-1) \times \ldots \times (k+1) \times (k-1) \times \ldots \times 3 \times 2) + 1]$.
Since each term in the sequence $n! + 2, n! + 3, \ldots n! + n$ is a product of two factors (other than 1 or itself), each term is a composite number for any value of n.

Section 4

Use the method of direct proof to prove the following statements.

Section 4

Use the method of direct proof to prove the following statements.
Q 28: If $a, b, c \in \mathcal{Z}$, $c \times gcd(a, b) \leq gcd(ca, cb)$

Section 4

Use the method of direct proof to prove the following statements.
Q 28: If $a, b, c \in \mathcal{Z}$, $c \times gcd(a, b) \leq gcd(ca, cb)$
Proof:

Section 4

Use the method of direct proof to prove the following statements.
Q 28: If $a, b, c \in \mathcal{Z}$, $c \times gcd(a, b) \leq gcd(ca, cb)$
Proof:
Suppose $gcd(a, b) = k$. Note that $gcd(a, b) \geq 0$ for any $a, b \in \mathcal{Z}$.
We consider the cases: $c = 0$ and $c \neq 0$.
Case 1: $(c = 0)$ In this case, $0.gcd(a, b)$ is equal to $gcd(0, 0) = 0$.
Case 2: $(c \neq 0)$
By the definition of divisibility, since k is a divisor of both a and b, we have $a = kx$ and $b = ky$ for integers x and y.
$ca = ckx$, $cb = cky$ thus the greatest divisor is at least $|ck|$.
Thus the result follows.
Consider an example where $a = 2, b = 4, c = -1$.
$gcd(2, 4) = gcd(-2, -4) = 2$,
$-1 * gcd(2, 4) = -2 < gcd(-2, -4) = 2$.