Counting II

- Topics not covered in the text
- Covered in section 6.5 of Rosen's book
- Covered in section 1.4 of Grimaldi's book

Permutations

- A **k-permutations** of a set of n objects is the same as a length-k lists. Here the order of the objects is important.
- The number of k-permutations of n objects with repetition is **n^k.**
- The number of k-permutation of n objects without repetition is $\frac{n!}{(n-k)!}$.
- We write $P(n,k) = \frac{n!}{(n-k)!}$

Permutations with indistinguishable objects

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  - If three Es and two Ss are treated as distinct, the number of 6-permutations is 6!.
  - Note that $JE^1S^1S^2E^2E^3$ and $JE^1S^2S^1E^2E^3$ are the same if two Ss are not distinct.
  - The number of distinct permutations is $\frac{6!}{3!2!}$.

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- **Second Method**: The word *JESSEE* contains 3 Es, 2 Ss and one J.

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  - After all these, J has one (C(1,1)) position to go.

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  - There are three more positions to fill once Es are placed.
  - In C(3,2) ways we can place 2Ss in three positions.
  - After all these, J has one (C(1,1)) position to go.
  - The total number of permutations of the letters of *JESSEE* is

$$C(6,3).C(3,2).C(1,1) = \frac{6!}{3!3!} \times \frac{3!}{2!1!} \times \frac{1!}{1!0!} = \frac{6!}{3!2!1!}.$$

Permutations with indistinguishable objects

- **Theorem**: The number of different permutations of n objects where there are n_1 objects of Type 1 (non-distinct), n_2 objects of Type 2 (non-distinct), …., n_t objects of Type t (non-distinct), and $\sum_{i=1}^{t} n_i = n$, is $\frac{n!}{n_1!n_2!.....n_t!}$.

Combinations

- A k-combination of a set of n objects is an unordered selection of k elements from the set. When the elements are not repeated, a k-combination is a size-k subset. We have seen that

$$C(n,k) = \frac{n!}{k!(n-k)!}.$$

Some important identities

$$C(n,k) = \frac{P(n,k)}{k!}$$

$$C(n,k) = C(n,n-k)$$

$$C(n+1,k) = C(n,k-1) + C(n,k)$$

Combinations with repetitions

- We consider the case when an object is selected repeatedly.
- Example: Consider a set A = {a,b,c,d}. We select two objects from A.
  - We now consider the following four cases.

- **4-permutations with repetitions**: $4^2 = 16$ possible cases.

aa	*ab*	*ac*	*ad*
ba	*bb*	*bc*	*bd*
ca	*cb*	*cc*	*cd*
da	*db*	*dc*	*dd*

- **4-permutations without repetitions**: P(4,2) = 12 possible cases.

	ab	*ac*	*ad*
ba		*bc*	*bd*
ca	*cb*		*cd*
da	*db*	*dc*	

- **4-combinations without repetitions**: C(4,2) = 6 possible cases.

	ab	*ac*	*ad*
		bc	*bd*
			cd

- **4-combinations with repetitions**: C(5,3) = 10 possible cases.

aa	*ab*	*ac*	*ad*
	bb	*bc*	*bd*
		cc	*cd*
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- **4-combinations with repetitions**: C(5,3) = 10 possible cases.

aa	*ab*	*ac*	*ad*
	bb	*bc*	*bd*
		cc	*cd*
			dd

Note that combinations with repetitions do not correspond to subsets of a set.

Combinations with repetitions

- Consider the following problem known as **distribution of money.**
  - We have n pennies that we want to distribute to k kids. Each child gets at least one penny. How many ways can we distribute the money?

Combinations with repetitions

- Consider the following problem known as **distribution of money.**
  - We have n pennies that we want to distribute to k kids. Each child gets at least one penny. How many ways can we distribute the money?
  - Kids are distinct, but pennies are not.
  - For n=6 and k=3, (1,1,4), (2,3,1) are distinct ways.
  - All solutions: (1,1,4), (1,2,3), (1,3,2), (1,4,1),(2,1,3), (2,2,2), (2,3,1), (3,1,2), (3,2,1), (4,1,1)

Combinations with repetitions

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  - There is only one way for kid 1 to get n_1 pennies, kid-2 to get n_2 pennies, ... and so on where

    $n_1 + n_2 + \ldots + n_k = n$

Distribution of money

- We have n pennies that we want to distribute to k kids. Each child gets at least one penny. How many ways can we distribute the money?
  - Let us consider the following experiment.
  - Line up the pennies (all are the same), order doesn't matter.

x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x

(n pennies)

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(n pennies)

  - Let the first child pick them from left to right. After a while we stop the kid.
  - **x x x x x x x x x x | x x x x x x x x x x x x x x x x x x x x x x**

(kid -1 gets these) (stop)

Distribution of money

– Line up the pennies (all are the same), order doesn't matter.

x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x

(n pennies)

– Let the first child pick them from left to right. After a while we stop the kid.

x x x x x x x x x x |x x x x x x x x x x x x x x x x x x x x x x

(kid -1 gets these) (stop)

Distribution of money

– Line up the pennies (all are the same), order doesn’t matter.

x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x

(n pennies)

– Let the first child pick them from left to right. After a while we stop the kid.

x x x x x x x x x x |x x x x x x x x x x x x x x x x x x x x x x

(kid -1 gets these) (stop)

– Let the second child pick pennies starting from where kid-1 stopped.

x x x x x x x x x x |x x x x x x x x x x x| x x x x x x x x x x x

(kid -1 gets these) (kid-2 gets these)

Distribution of money

– Line up the pennies (all are the same), order doesn't matter.

x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x

(n pennies)

– The distribution of money is determined by specifying where to start with a new child.
– The first child starts from 1.
– The other k-1 kids can enter at position 2, 3, 4, …, n-1.
– This means that there are C(n-1,k-1) ways to chose an entry point.

Distribution of money

- There are C(n-1,k-1) ways to distribute n pennies to k kids with the constraint that each kid gets at least one penny.

Distribution of money (no restriction)

- Problem: Distribute n pennies to k kids with no restriction on whether a kid gets a penny or not.
- We use the following trick:
  - We borrow one penny from each kid, and then distribute (n+k) pennies to k kids such that each kid gets at least one penny.
- There are C(n+k-1,k-1) ways to distribute n pennies to k kids with no restriction.

Combinations with repetitions

- Given n distinct objects, select k objects where repetitions are allowed and the order of selecting objects is not important. (Note that k could be larger than n.)
- Here kids are n objects, and pennies are the k objects to be selected.
- The number of possible k-combinations is C(#kids + #pennies -1, #kids -1) = C(k+n-1, n-1)

Permutations and combinations with and without repetitions.

Type	Repetition Allowed?	Formula
k-permutations	No	$P(n,k) = \frac{n!}{(n-k)!}$
k-permutations	Yes	n^k
k-combinations	No	$C(n,k) = \frac{n!}{k!(n-k)!}$
k-combinations	Yes	$C(n+k-1,n-1) = \frac{(n+k-1)!}{(n-1)!k!}$

Number of integer solutions

Problem: Let x_i , $1 \le i \le n$ be n nonnegative integer variables. Determine all integer solutions to the equation

$$x_1 + x_2 + \ldots + x_n = m \text{ where } x_i \ge 0 \text{ for all } 1 \le i \le n.$$

- Suppose n=4, and m=7
  - $x_1=3, x_2=3, x_3=0, x_4=1$ is one solution to the equation.
  - $x_1=1, x_2=0, x_3=3, x_4=3$ is another different solution.
  - A possible interpretation for the solution $x_1=3, x_2=3, x_3=0, x_4=1$ is that we are distributing 7 pennies (identical) among 4 kids (distinct). We have given 3 pennies each to kid 1 and kid 2, nothing to the third kid, and kid 4 gets one penny.

Number of integer solutions

Problem: Let x_i , $1 \le i \le n$ be n nonnegative integer variables. Determine all integer solutions to the equation

$$x_1 + x_2 + \ldots + x_n = m \text{ where } x_i \ge 0 \text{ for all } 1 \le i \le n.$$

- m = # of pennies; n = # of kids; no restriction (i.e. a kid can get no penny)
- Total number of integral solutions = C(n+m-1, n-1)

It is important to recognize the equivalence of the following

- The number of integer solutions of the equation $x_1 + x_2 + \ldots + x_n = m$ where $x_i \geq 0$ for all $1 \leq i \leq n$.
- The number of choices, with repetitions, of size m from a collection of n objects.
- The number of choices of distributing m pennies to n kids with no restriction (i.e. a kid can get zero penny).
- The number of ways of placing m balls in n distinct bins.

Example

- A doughnut shop has plain doughnuts, cherry doughnuts, chocolate doughnuts, almond doughnuts, apple doughnuts, broccoli doughnuts. How many ways are there to choose:
  (a) a dozen doughnuts?

Example

- A doughnut shop has plain doughnuts, cherry doughnuts, chocolate doughnuts, almond doughnuts, apple doughnuts, broccoli doughnuts. How many ways are there to choose:
  (a) a dozen doughnuts?

  12 indistinguishable balls and 6 bins, or 12 pennies and 6 kids

  Ans: C(6+12-1,6-1) = C(17,5) = C(17, 12)

Example

- A doughnut shop has plain doughnuts, cherry doughnuts, chocolate doughnuts, almond doughnuts, apple doughnuts, broccoli doughnuts. How many ways are there to choose:
  (b) three dozen doughnuts?

Example

- A doughnut shop has plain doughnuts, cherry doughnuts, chocolate doughnuts, almond doughnuts, apple doughnuts, broccoli doughnuts. How many ways are there to choose:

  (b) three dozen doughnuts? 36 doughnut

  36 indistinguishable balls and 6 bins, or 36 pennies and 6 kids

  Ans: C(6+36-1,6-1) = C(41,5) = C(41,36)

Example

- A doughnut shop has plain doughnuts, cherry doughnuts, chocolate doughnuts, almond doughnuts, apple doughnuts, broccoli doughnuts. How many ways are there to choose:
  - (c) two dozen doughnuts with at least two of each kind?

Example

- A doughnut shop has plain doughnuts, cherry doughnuts, chocolate doughnuts, almond doughnuts, apple doughnuts, broccoli doughnuts. How many ways are there to choose:

  (c) two dozen doughnuts with at least two of each kind?

Pick first two of each kind . Thus the number is the number of ways of choosing the remaining dozen.

Same as question (a).

Example

- A doughnut shop has plain doughnuts, cherry doughnuts, chocolate doughnuts, almond doughnuts, apple doughnuts, broccoli doughnuts. How many ways are there to choose:
  - (d) two dozen doughnuts with no more than two broccoli doughnuts?

Example

- A doughnut shop has plain doughnuts, cherry doughnuts, chocolate doughnuts, almond doughnuts, apple doughnuts, broccoli doughnuts. How many ways are there to choose:

  (d) two dozen doughnuts with no more than two broccoli doughnuts?

  We will add up three cases: no broccoli doughnut, exactly one broccoli doughnut, exactly two broccoli doughnuts.

  These numbers are: C(5 + 24 -1, 5-1) (0 broccoli doughnut);

  C(5+23-1, 5-1) (1 broccoli doughnut); C(5+22-1,5-1) (2 broccoli doughnuts)

Example

- A doughnut shop has plain doughnuts, cherry doughnuts, chocolate doughnuts, almond doughnuts, apple doughnuts, broccoli doughnuts. How many ways are there to choose:

  (e) two dozen doughnuts at least five chocolate doughnuts and at least three almond doughnuts?

Example

- A doughnut shop has plain doughnuts, cherry doughnuts, chocolate doughnuts, almond doughnuts, apple doughnuts, broccoli doughnuts. How many ways are there to choose:

  (e) two dozen doughnuts at least five chocolate doughnuts and at least three almond doughnuts?

  We have already chosen the first 8, so need to select the remaining 16. There are C(6+16-1,6-1) ways to do this.

Example

- A doughnut shop has plain doughnuts, cherry doughnuts, chocolate doughnuts, almond doughnuts, apple doughnuts, broccoli doughnuts. How many ways are there to choose:

    (e) two dozen doughnuts with at least one plain, at least two cherry, at least three chocolate, at least one almond, at least two apple, no more than three broccoli doughnuts?

Example

- A doughnut shop has plain doughnuts, cherry doughnuts, chocolate doughnuts, almond doughnuts, apple doughnuts, broccoli doughnuts. How many ways are there to choose:

  (e) two dozen doughnuts with at least one plain, at least two cherry, at least three chocolate, at least one almond, at least two apple, no more than three broccoli doughnuts?

  We have already chosen the first nine doughnuts. We need determine the ways to distribute 15 doughnuts without choosing more than 3 broccoli doughnuts. The answer is

  C(5+15-1,5-1) + C(5+14-1,5-1) + C(5+13-1,5-1) + C(5+12-1, 5-1).

Example

- In bridge, the 52 cards are dealt to 4 players. How many different ways to deal bridge hands?

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  – The answer is: C(52,13)*C(39,13)*C(26,13)*C(13,13)

Example

- How many terms are there in the expansion of $(w + x + y + z)^{100}$?

Example

- How many terms are there in the expansion of $(w + x + y + z)^{100}$?
  - each term of in the expansion of $(w + x + y + z)^{100}$ is of the form $w^a x^b y^c z^d$ where $a + b + c + d = 100$ where a, b, c, d are nonnegative integers.
  - Therefore, the answer is C(4 + 100 -1, 4 – 1).

Example: Consider the following programming piece

```
for i= 1 to 100 do
   for j= 1 to i do
      for k= 1 to j do
         print(i*j+k)
```

How many times the print statement is executed?

Example: Consider the following programming piece

```
for i= 1 to 100 do
   for j= 1 to i do
      for k= 1 to j do
         print(i*j+k)
```

How many times the print statement is executed?

This is equivalent to selecting three integers from the set {1,2, ..., 100} with repetitions. (Here pennies = 3; kids = 100)

The answer is C(3+100-1,100-1) = C(102,99).

Example

- Find the number of non-negative integer solutions to

  $x_1 + x_2 + x_3 = 20$, $x_1 \leq 6$, $x_2 \leq 7$, $x_3 \leq 8$.

Example

- Find the number of non-negative integer solutions to
  $x_1 + x_2 + x_3 = 20$, $x_1 \le 6$, $x_2 \le 7$, $x_3 \le 8$.
  - U: set of integer solution with $x_i \ge 0$, for all i.
  - A: a set of integer solutions with $x_1 \ge 7$.
  - B: a set of integer solution with $x_2 \ge 8$.
  - C: a set of integer solution with $x_3 \ge 9$.
  - $A \cap B$: a set of integer solution with $x_1 \ge 7$, $x_2 \ge 8$.
  - $A \cap C$: a set of integer solution with $x_1 \ge 7$, $x_3 \ge 9$.
  - $B \cap C$: a set of integer solution with $x_2 \ge 8$, $x_3 \ge 9$.
  - $A \cap B \cap C$: a set of integer solution with $x_1 \ge 7$, $x_2 \ge 8$, $x_3 \ge 9$.
- **Answer: $|U| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C|$ $-|A \cap B \cap C|$.**

Example

- Find the number of non-negative integer solutions to

  $x_1 + x_2 + x_3 = 20,\ x_1 \le 6,\ x_2 \le 7,\ x_3 \le 8.$

  - U: set of integer solution with $x_i \ge 0$, for all i. **$|U| = C(3+20-1,3-1)$**
  - A: a set of integer solutions with $x_1 \ge 7$. **$|A| = C(3+13-1,3-1)$**
  - B: a set of integer solution with $x_2 \ge 8$. **$|B| = C(3+12-1, 3-1)$**
  - C: a set of integer solution with $x_3 \ge 9$. **$|C| = C(3+11-1,3-1)$**
  - $A \cap B$: a set of integer solution with $x_1 \ge 7$, $x_2 \ge 8$. **$|A \cap B| = C(3+5-1,2)$**
  - $A \cap C$: a set of integer solution with $x_1 \ge 7$, $x_3 \ge 9$. **$|A \cap C| = C(3+4-1,2)$**
  - $B \cap C$: a set of integer solution with $x_2 \ge 8$, $x_3 \ge 9$. **$|B \cap C| = C(3+3-1,2)$**
  - $A \cap B \cap C$: a set of integer solution with $x_1 \ge 7$, $x_2 \ge 8$, $x_3 \ge 9$. **Empty**
- **Answer: $|U| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C| = 3$**

Example

Determine the number of integer solutions to the following:

(a) equation $x_1 + x_2 + x_3 = 6, x_i \geq 0, 1 \leq i \leq 3$.

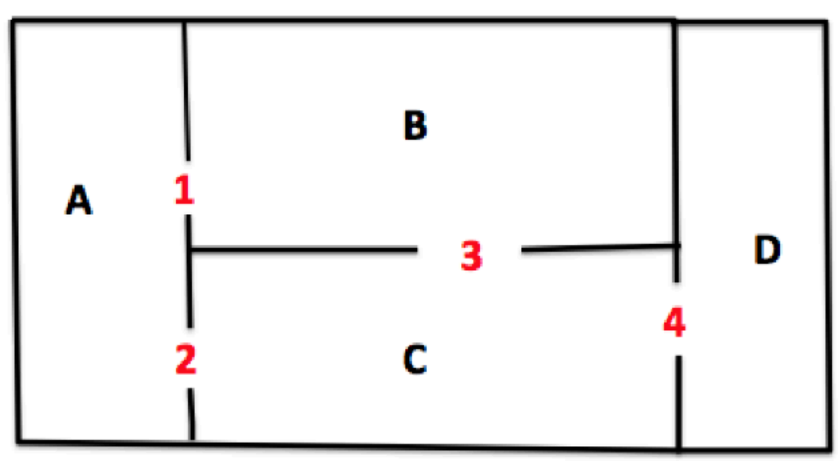
(b) equation $x_1 + x_2 + x_3 + x_4 + x_5 = 15, x_i \geq 0, 1 \leq i \leq 5$.

(c) equations $x_1 + x_2 + x_3 = 6$ and $x_1 + x_2 + x_3 + x_4 + x_5 = 15, x_i \geq 0, \quad 1 \leq i \leq 5$.

(d) equations $x_1 + x_2 + x_3 \leq 6$ and $x_1 + x_2 + x_3 + x_4 + x_5 \leq 15, x_i \geq 0, \quad 1 \leq i \leq 5$.

Example

***Four connecting rooms are to be painted with *k* district colors so that no two adjacent rooms have the same color. Room A is connected to room B (by door 1) and room C (by door 2). Room B is connected to room C (by door 3). Room C is connected to room D by door 4. How many ways can one paint the rooms so that no two adjacent rooms (shared by a door) cave the same colour.



Example

{ Let D_i be the painting schemes if the wall containing door i is removed. $D_i \cap D_j$ is set of the painting schemes coloring the rooms if the walls containing i and j are removed. We can define similarly $D_i \cap D_j \cap D_k$ and $D_i \cap D_j \cap D_k \cap D_l$. Note that when all 4 walls containing doors are removed, there are only k ways to paint one room. If there is no constraint on colors of adjacent rooms, k^4 different ways we can paint four rooms. }