Counting

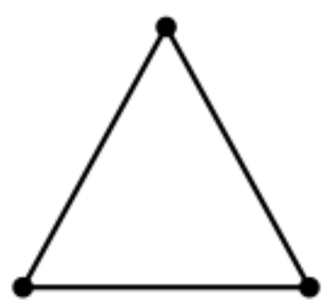
Chapter 3 of the text

A few slides have been taken from the sites

http://cse.unl.edu/~choueiry/S13-235/

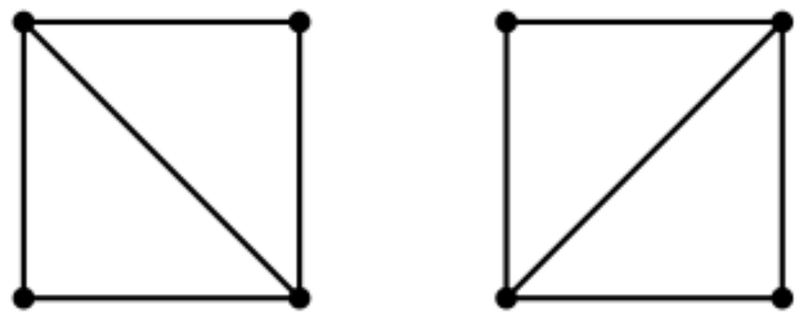
Combinatorics

- Combinatorics is the study of arrangement of objects. For example:
  - In how many ways can a convex polygon with n sides be divided into triangles by adding non-crossing diagonals?
- For n=3, there is only one.



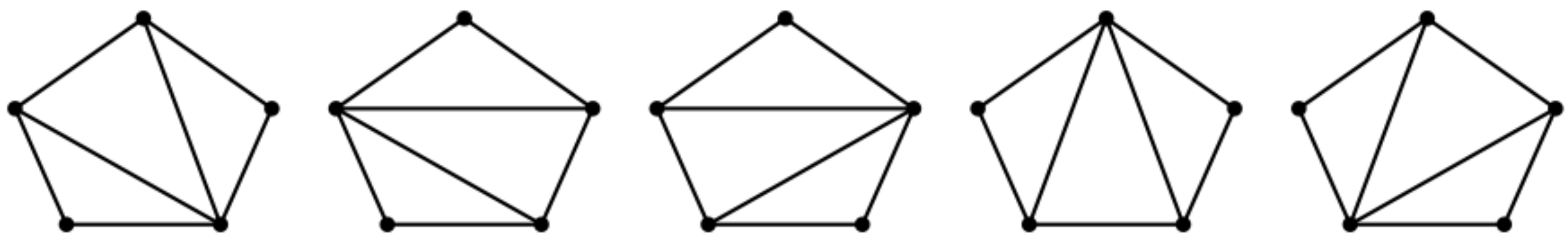
Combinatorics

- For n=4, there are 2 such arrangements.



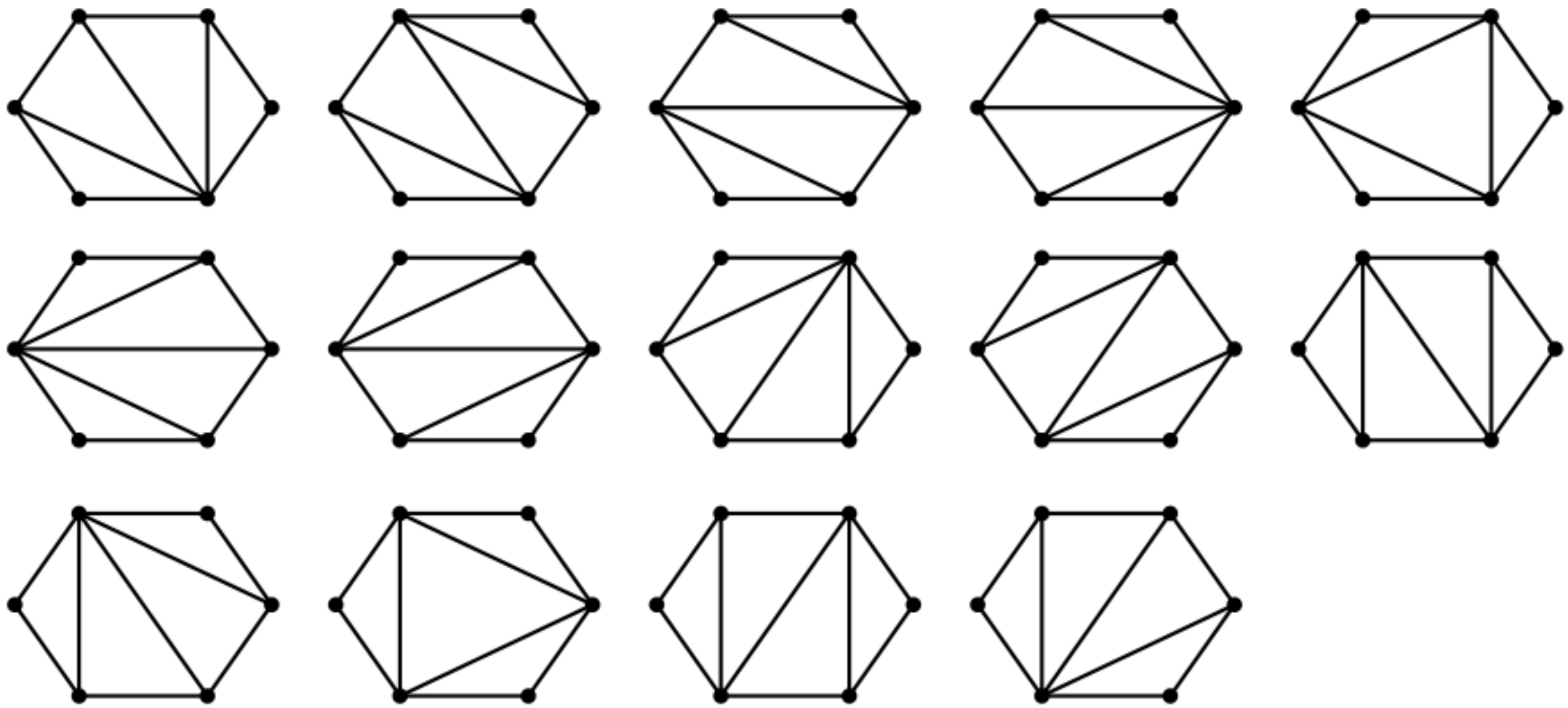
Combinatorics

- For n=5, there are 5 such arrangements.



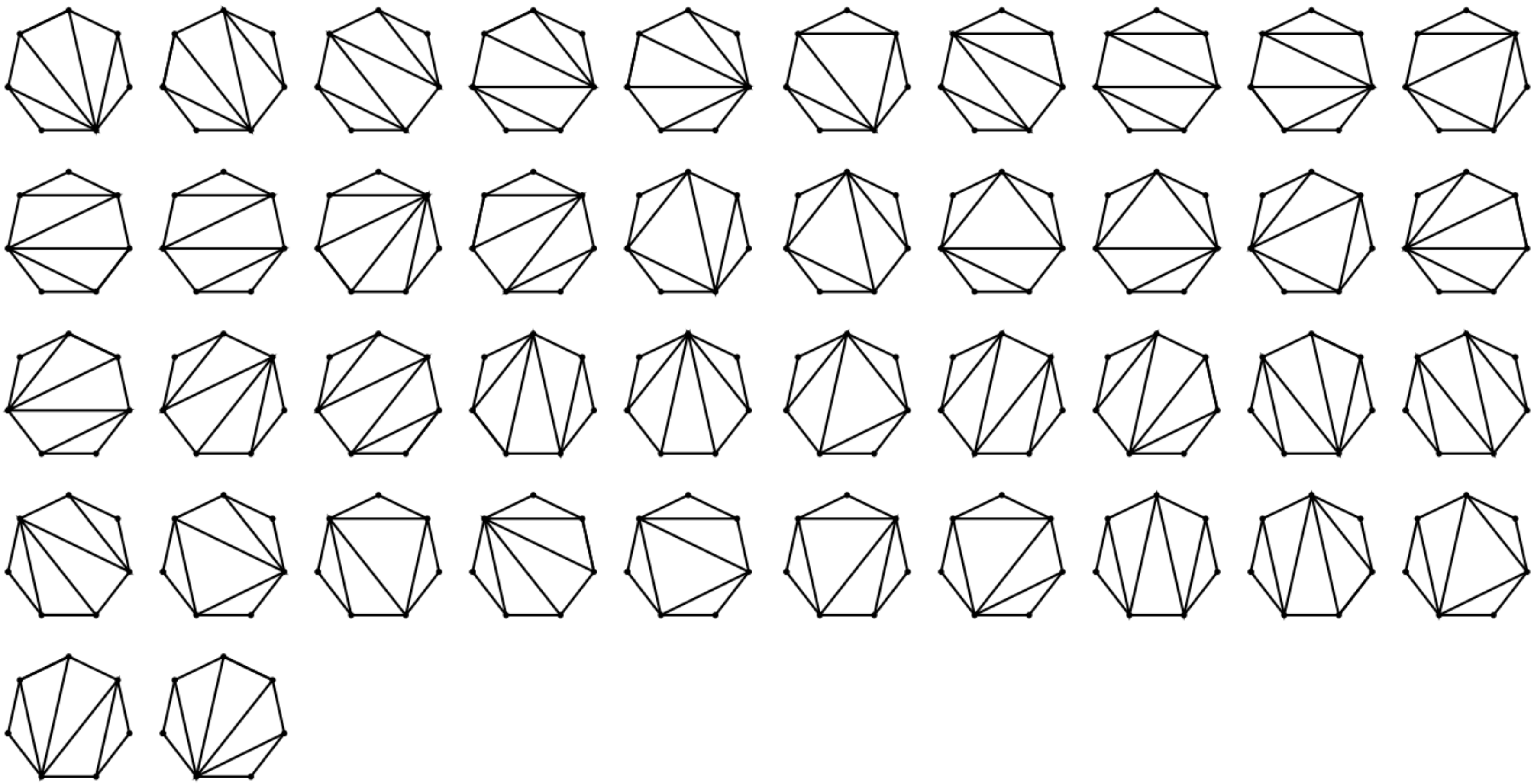
Combinatorics

- For n=6, there are 14 such arrangements.



Combinatorics

- For n=7, there are 42 such arrangements.



Combinatorics

- Combinatorics is the study of arrangement of objects.
- Counting of objects with certain properties is an important component of combinatorics.
- We use counting to determine the complexity (running time) of algorithms.
- Counting plays important role in determining probabilities of events.

Counting

- In the class, the topic will be covered along the following lines.
  - First go through Chapter 3 of the text.
  - Next we will cover the materials of Chapter 6 of the book by Rosen. We will only cover the parts of this chapter not covered in the text.

Counting Lists (Section 3.1)

- A list is an ordered sequence of objects.
  - Top 10 movies: A list of 10 movies which can be represented as (name 1, name 2, …, name 10).
  - The objects are ordered.
  - i^{th} element of the list is the i^{th} object.
  - (a,b) list different than (b,a).
  - The number of elements in the list is called its length.
  - Note the similarity with n-tuple we discussed earlier.

Counting Lists (Section 3.1)

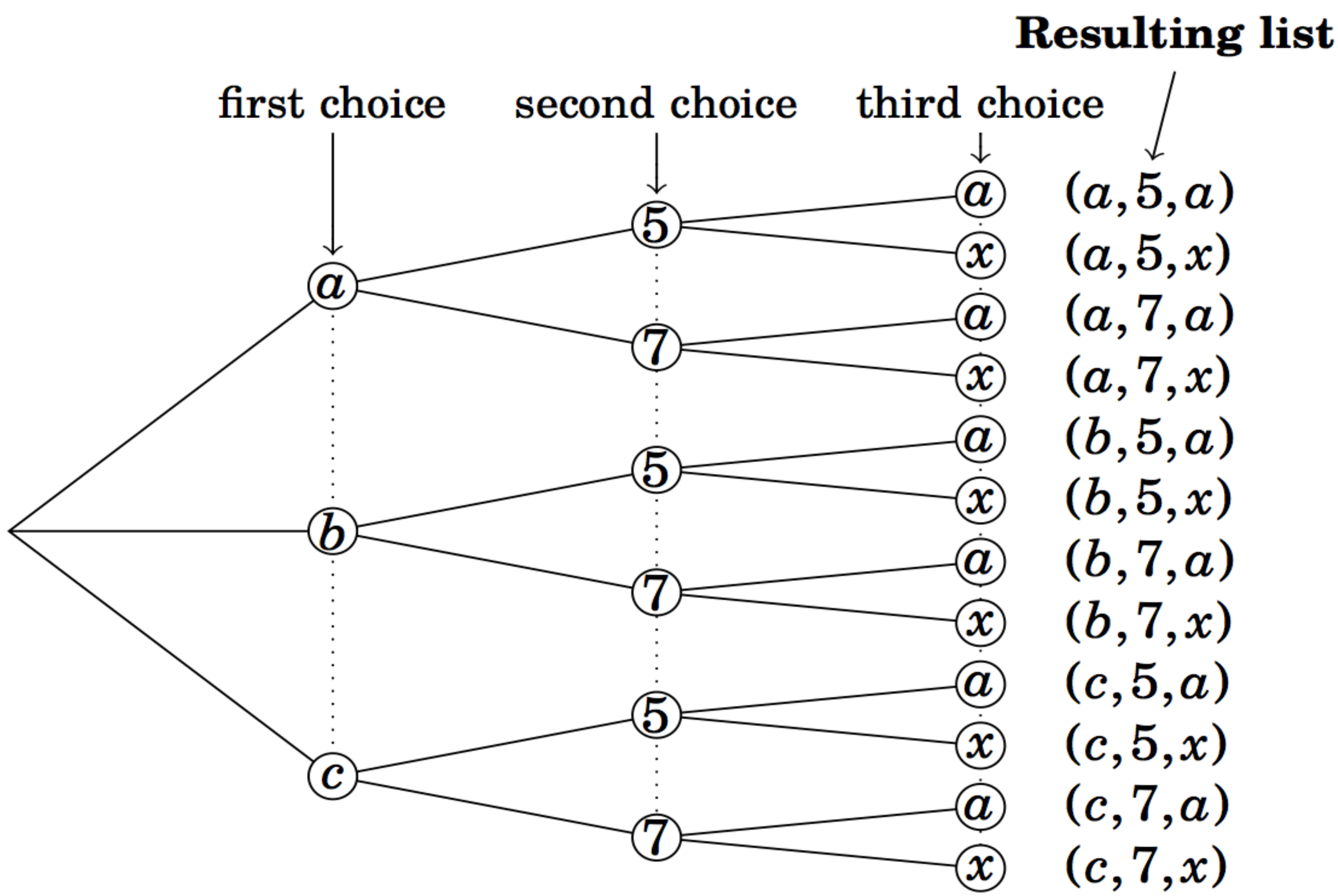
- Consider two sets:
  - A = MACM 101 students in the class;
    |A| = # of students
  - G = Set of possible grades ; |G| is the # of grades.
- Consider a list with two objects: (name, grade) where the first object is name, an element of A, and the second object is grade, an element of G.
- How many possible different 2-tuples are possible?
- Note that it is nothing but the size of A x G.

Counting Lists (2)

- Suppose we want to make a list of length three having the property that
  - the first entry must be an element of the set A = {a,b,c},
  - the second entry must be in B= {5,7} and
  - the third entry must be in C={a,x}.
- How many such lists altogether?
- Again it is |A x B x C|.
- How do we generate these lists systematically?

Counting Lists

- Constructing lists of length three:



Multiplication Principle

- In making a list of length n, suppose there are a_1 possible choices for first entry, a_2 possible choices for the second entry, a_3 possible choices for the third entry, and so on. Then the total number of different lists that can be made is the product $a_1 \times a_2 \times a_3 \times \ldots \times a_n$.

Example

- In BC, the license plate of a car has either three letters followed by three digits, or 3 digits followed by three letters. For example ABC007, 007ABC are two standard license plates. How many different license plates are possible?
  - Note that any license plates such as ABC007 corresponds to a length-6 list (A, B, C, 0, 0, 7).
  - # of license plates of the type (letter,letter,letter, digit,digit,digit) is 26 x 26 x 26 x 10 x 10 x 10

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  - # of license plates of the type (letter,letter,letter, digit,digit,digit) is 26 x 26 x 26 x 10 x 10 x 10
  - # of license plates of the type (digit,digit,digit,letter,letter,letter) is 10 x 10 x 10 x 26 x 26 x 26

Example

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  - # of license plates of the type (letter,letter,letter, digit,digit,digit) is 26 x 26 x 26 x 10 x 10 x 10
  - # of license plates of the type (digit,digit,digit,letter,letter,letter) is 10 x 10 x 10 x 26 x 26 x 26
  - **Total = 2 x 26 x 26 x 26 x 10 x 10 x 10**

Example

- Consider making lists from the symbols A, B, C, D, E, F, G

(a) How many length-4 lists are possible if repetition is allowed (i.e. a symbol may appear more than once)?

(b) How many length-4 lists are possible if repetition is **not** allowed?

(c) How many length-4 lists are possible if repetition is **not** allowed and the list must contain an E?

(d) How many length-4 lists are possible if repetition is allowed and the list must contain an E?

Example

- Consider making lists from the symbols A, B, C, D, E, F, G

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  Ans:

Example

- Consider making lists from the symbols A, B, C, D, E, F, G

  (a) How many length-4 lists are possible if repetition is allowed (i.e. a symbol may appear more than once)?

  Ans:

  There are 7 choices for each position of the list. Therefore, the number of length-4 lists in this case is
  7 x 7 x 7 x 7.

Example

- Consider making lists from the symbols A, B, C, D, E, F, G

(b) How many length-4 lists are possible if repetition is **not** allowed?

Ans

Example

- Consider making lists from the symbols A, B, C, D, E, F, G

(b) How many length-4 lists are possible if repetition is **not** allowed?

Ans

- Here repetition is not allowed.
- Note that once the letter for position one of the list is chosen, the same letter cannot be chosen again.
- Thus the choice for the first position is 7, and the choice for the second position is 6.
- Once the second position of the list is filled, there only 5 choices for the third position of the list.
- Therefore, the total number length-4 lists is 7 x 6 x 5 x 4

Example

- Consider making lists from the symbols A, B, C, D, E, F, G

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- Consider making lists from the symbols A, B, C, D, E, F, G

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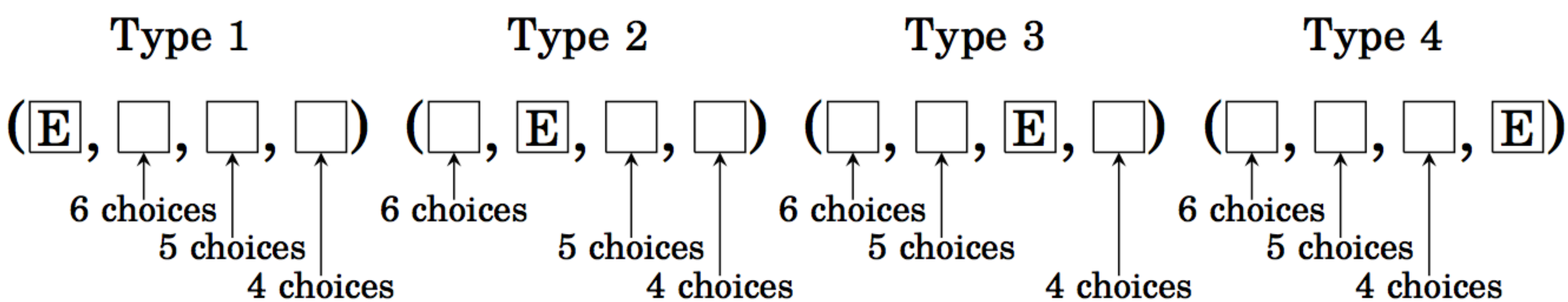
Ans: There are four types of lists depending on whether E occurs as the first, second, third or fourth entry. These four types are shown below.



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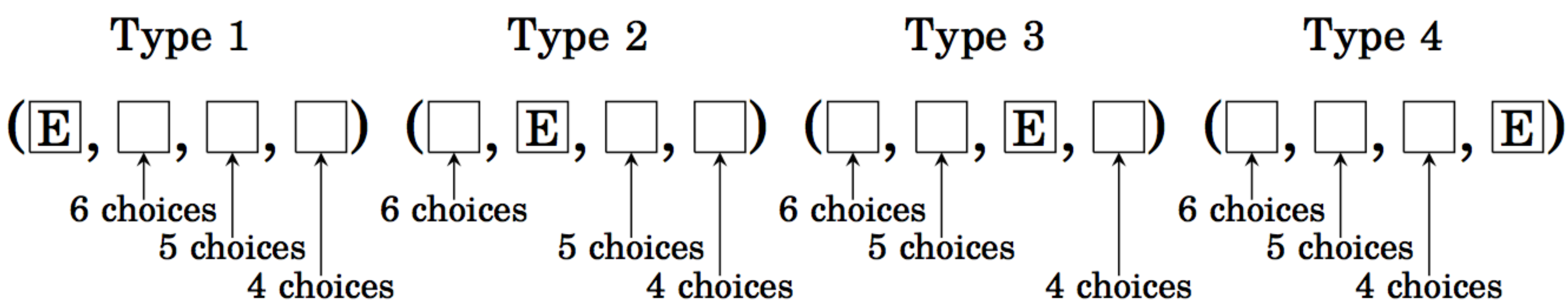

Total # =(6 x 5 x 4) + (6 x 5 x 4) + (6 x 5 x 4) + (6 x 5 x 4)

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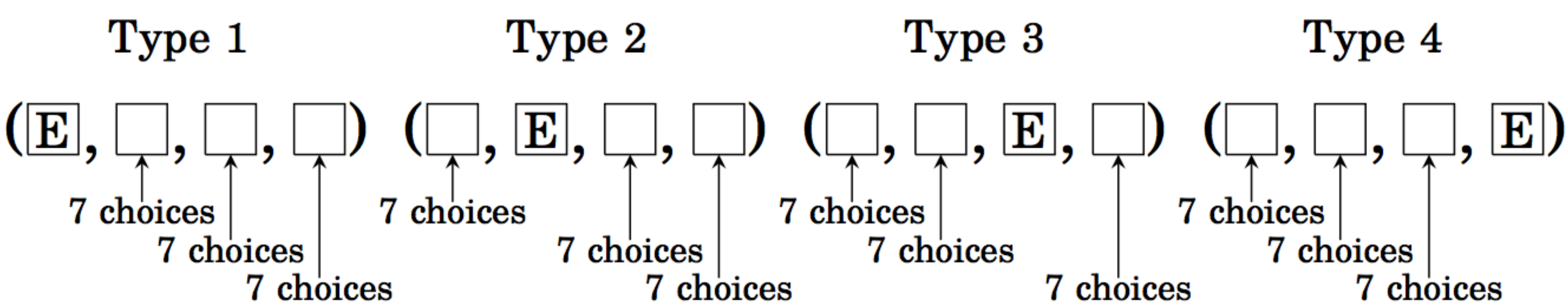

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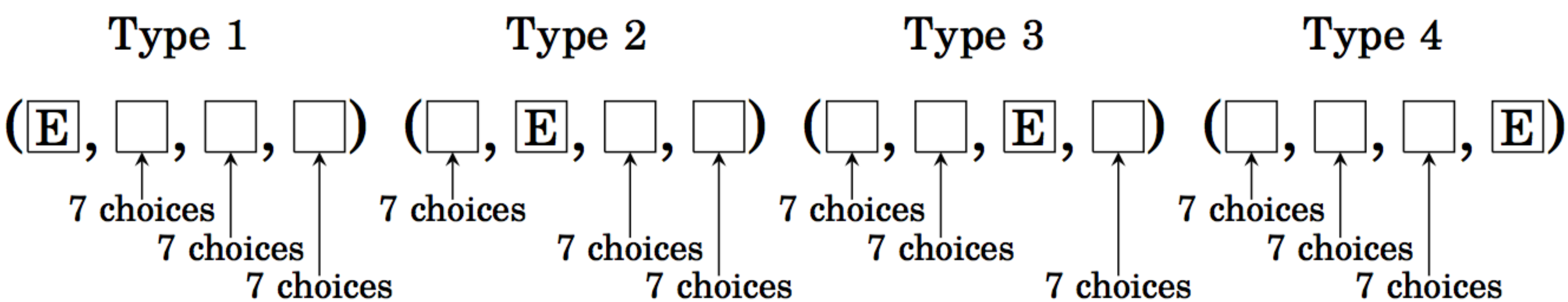

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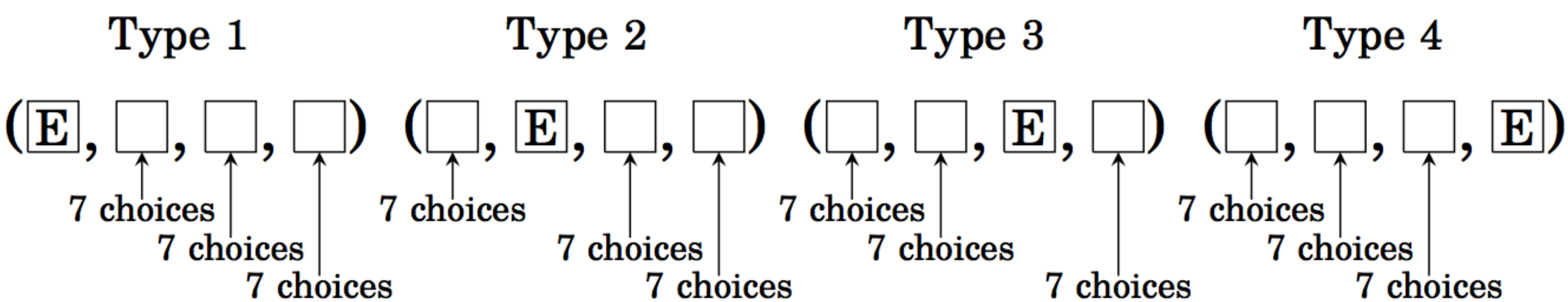

larger than the correct value of 1105.

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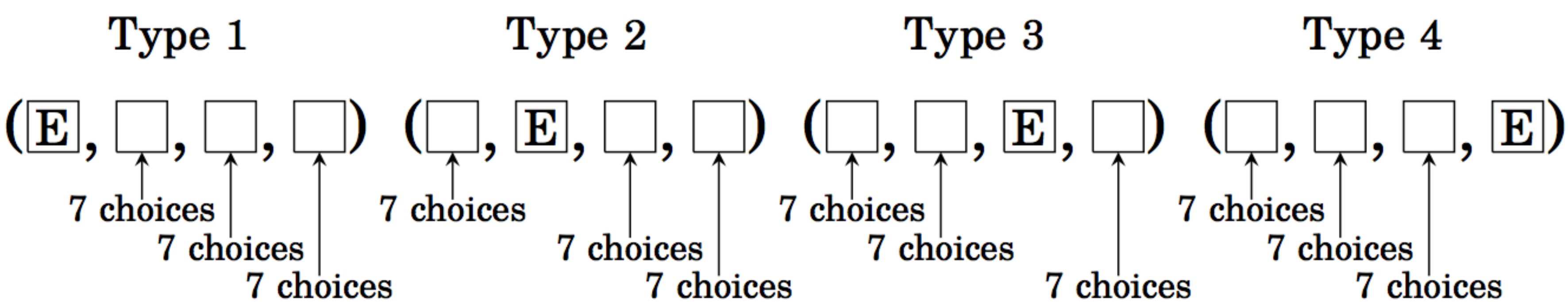

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Note: The list (E,A,E,C,D) is counted twice, one as a type 1 and one as a type 3. Similarly (E,E,E,E) is counted 4 times.

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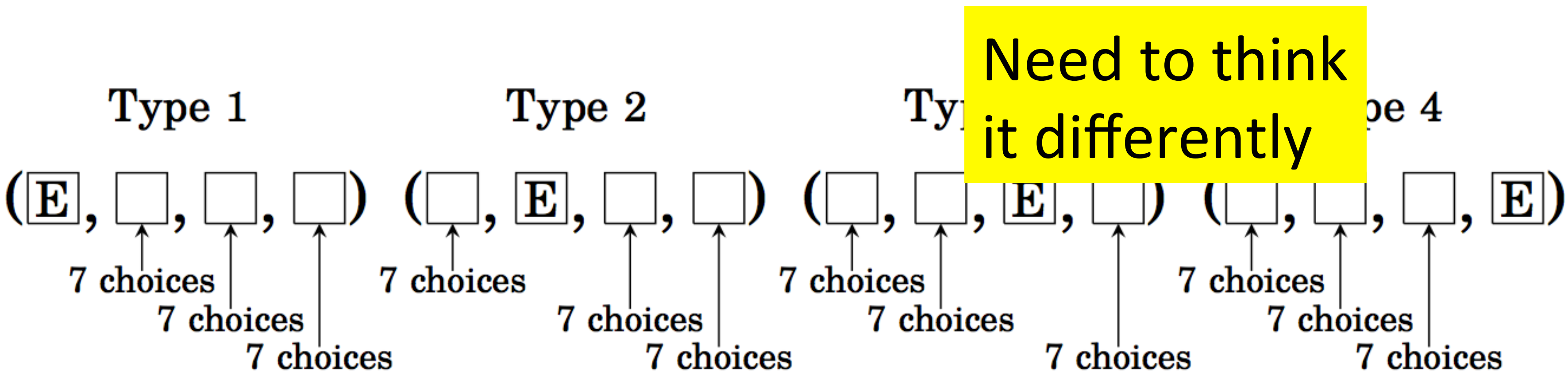

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  Ans: Correct thinking:

  (a) We know that the number of length-4 lists with repetitions is 7^4.

Example

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  (d) How many length-4 lists are possible if repetition is allowed and the list must contain an E?

  Ans: Correct thinking:

  (a) We know that the number of length-4 lists with repetitions is 7^4.

  (b) There are many lists which contain no E. We subtract these lists (containing no E) from 7^4 to obtain the number of lists that contain at least one E.

  There are 6^4 lists that do not have an E.

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(c) Therefore there are $7^4 – 6^4$ = 1105 lists with repetition allowed that contain at least one E.

In solving counting problems, we must always be careful to avoid this kind of multiple counting.

Examples from text

- 3.1(4) Five cards are dealt off of a standard 52-card deck and lined up in a row. How many such line ups are there in which all 5 cards are of the same suit?
- **Ans**:

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- **Ans**: There are 4 suites: club, diamond, heart and spade. Each suite has 13 cards.
  - how many rows (lists) of 5-card club suite?

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- **Ans**: There are 4 suites: club, diamond, heart and spade. Each suite has 13 cards.
  - how many rows (lists) of 5-card club suite?
    - first element has 13 choices
    - second element has 12 choices
    - third element has 11 choices
    - fourth element has 10 choices
    - fifth element has 9 choices.

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  - how many rows (lists) of 5-card club suite?
    - first element has 13 choices
    - second element has 12 choices
    - third element has 11 choices
    - fourth element has 10 choices
    - fifth element has 9 choices.
  - number of 5-card lists of club suites = 13.12.11.10.9
  - number of 5-card lists of four suites= 4.13.12.11.10.9

Examples from text

- 3.1(10) This problem concerns list made from the letters A, B, C, D, E, F, G, H, I, J.
  (a) How many length-5 lists can be made from these letters if repetition is not allowed, and the list must begin with a vowel?
  (b) How many length-5 lists can be made from these letters if repetition is not allowed, and the list must begin and end with a vowel?
  (c) How many length-5 lists can be made from these letters if repetition is not allowed, and the list must contain exactly one A.

Examples from text

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  Ans: There are 3 vowels in the given letters: A, E and I.

  - There are three types of lists we need to count. One type that starts with A, the other two types that start with E and I.
  - Lists that start with A: Since no repetition is allowed, there are 9 choices for position 2, 8 choices for position 3, 7 choices for position 4 and 6 choices for position 5.

    Total number of lists for this type: 9.8.7.6
  - Total number of lists of all types: 3x(9.8.7.6)

Examples from text

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  (b) How many length-5 lists can be made from these letters if repetition is not allowed, and the list must begin and end with a vowel?

  Ans: There are six types lists: one starts with A and ends with E.; one starts with A and ends in I; one starts with E and ends with A; one starts with E and ends with I; one starts with I and ends with A; and one starts with I and ends with E.

  - Lists that starts with A and ends in E: the choices for positions 2, 3, 4 are 8, 7, 6 respectively. Total such lists is 8.7.6.
  - Total number of lists of all types is 6x(8.7.6)

Note that 6 types of lists come from the fact there are 3 vowels enumerating lists with 2 positions.

Examples from text

- 3.1(10) This problem concerns list made from the letters A, B, C, D, E, F, G, H, I, J.
  - (c) How many length-5 lists can be made from these letters if repetition is not allowed, and the list must contain exactly one A.

    Very similar to a problem solved earlier.

Factorial (section 3.2)

- For any positive integer n, **n!** means:
  - **n(n-1)(n-2) ……. 3.2.1**
- **0!** is defined as equal to **one**.
- Examples: 4! = 4.3.2.1 = 24.
- Examples: 3.5! = 3.(5!)

Example

- Consider the problem of counting lists of length seven from the symbols 0, 1, 2, 3, 4, 5, 6.

  (a) How many such lists if repetition is not allowed

  Ans: 7.6.5.4.3.2.1

Example

- Consider the problem of counting lists of length seven from the symbols 0, 1, 2, 3, 4, 5, 6.

  (a) How many such lists if repetition is not allowed

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  (a) How many such lists if repetition is not allowed
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  (b) How many such lists if repetition is not allowed, and the first three entries must be odd.

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  (a) How many such lists if repetition is not allowed

  Ans: 7.6.5.4.3.2.1 which is 7!.

  (b) How many such lists if repetition is not allowed, and the first three entries must be odd.

  - There are three odd numbers in the seven symbols. We therefore fill first three positions with odd numbers and the last four positions with even numbers. Thus the total number is 3.2.1.4.3.2.1 which is 3!4!. Thus we note that there 3! ways to form length-3 lists and 4! ways to form length-4 lists.

Example

- Consider the problem of counting lists of length seven from the symbols 0, 1, 2, 3, 4, 5, 6

(c) How many such lists are there in which repetition is allowed, and the list must contain at least one repeated number.

Ans: $7^7 - 7!$

Why?

Theorem

- The number of non-repetitive lists of length-k whose entries are chosen from a set of n possible entries is $\frac{n!}{(n-k)!}$.

Proof: Using the multiplication rule we see that the number of length-k lists is n.(n-1).(n-2). .... (n-k+1).

We can rewrite n.(n-1).(n-2). .... (n-k+1) this as

$$\frac{n(n-1)(n-2)\cdots(n-k+1)(n-k)(n-k-1)\cdots 3\cdot 2\cdot 1}{(n-k)(n-k-1)\cdots 3\cdot 2\cdot 1} = \frac{n!}{(n-k)!}.$$

Examples

- 3.2(4) Using only pencil and paper, find the value of $\frac{100!}{95!}$.
- 3.2(6) There are two 0’s at the end of 10!=3,268,800. Using only paper and pencil, determine how many 0’s are at the end of 100!.
- 3.2(8) Compute how many 7-digit numbers can be made from the digits 1,2,3,4,5,6,7 if there is no repetition and the odd digits must appear in an unbroken sequence. (3571264, 2413576 are ok, but not 7234615)

Solution

- 3.2(4) Using only pencil and paper, find the value of $\frac{100!}{95!}$.

  Easy.

- 3.2(6) There are two 0's at the end of 10!=3,268,800. Using only paper and pencil, determine how many 0's are at the end of 100!.

Solution

- 3.2(4) Using only pencil and paper, find the value of $\frac{100!}{95!}$.
  Easy.
- 3.2(6) There are two 0's at the end of 10!=3,268,800. Using only paper and pencil, determine how many 0's are at the end od 100!.

  Ans: The two zeros in 10! come from the fact that
  - (1) The number is divisible by 10, and
  - (2) The number is divisible by both 5 and 2.

  Now 100! is divisible by 100*(95*92)* 90*(85*82)* .....10*(5*2).
  - The number of trailing zeros in 100! is 21.

Solution

- 3.2(8) Compute how many 7-digit numbers can be made from the digits 1,2,3,4,5,6,7 if there is no repetition and the odd digits must appear in an unbroken sequence. (3571264, 2413576 are valid, but not 7234615)
- Ans:

Solution

- 3.2(8) Compute how many 7-digit numbers can be made from the digits 1,2,3,4,5,6,7 if there is no repetition and the odd digits must appear in an unbroken sequence. (3571264, 2413576 are ok, but not 7234615)
- Ans:
  - There are 4 odd digits. There are 4! length-4 lists of odd integers without repetition.
  - The 7-digit valid number can be visualized as a list of length 4 using symbols 2, 4, 6, and a length-4 list of odd integers. There are 4! different such lists.
  - The total number of valid 7-digit sequence is $(4!)^2$.

Counting Subsets (section 3.3)

- In counting the number of length-k lists from a set of n possible objects, the order of the objects were very important.
- What happens when the order is not important?
- Now we select size-k subsets from a set of n possible objects.
- How many such size-k subsets are there?

Counting Subsets (section 3.3)

- In counting the number of length-k lists from a set of n possible objects, the order of the objects were very important.
- What happens when the order is not important?
- Now we select size-k subsets from a set of n possible objects.
- How many such size-k subsets are there?
- Suppose A = {a,b,c,d,e}
- Size-2 subsets of A are: {a,b}, {a,c}, {a,d}, {a,e}, {b,c}, {b,d}, {b,e}, {c,d}, {c,e}, {d,e}.
- Observe length-2 lists: (a,b), (b,a), (a,c), (c,a), (a,d), (d,a), (a,e), (e,a), (b,c), (c,b), (b,d), (d,b), (b,e), (e,b), (c,d), (d,c), (c,e), (e,c), (d,e), (e,d)

Counting Subsets

- Suppose A = {a,b,c,d,e}
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- There is definitely some connection between the number of size-k subsets and the number of length-k lists.

Counting Subsets

- Consider a size-k subset $\{a_1, a_2, \ldots, a_k\}$.
- This size-k subsets generate k! length-k lists using the k symbols $a_1, a_2, \ldots, a_k$.
- These k! lists are present in $\frac{n!}{(n-k)!}$. length-k lists using n symbols.
- Thus every size-k subset realizes k! length-k lists.
- Therefore,
  - number of size-k subsets x k! = number of length-k lists

$\Longrightarrow$ number of size-k subsets = C(n,k) = $\frac{n!}{k!(n-k)!}$

Counting Subsets

- In the text, C(n,k) is written as $\binom{n}{k}$.
- $\binom{n}{k}$ is read as ``**n choose k**''
- C(n,k) denotes the number of subsets that can be made by choosing k elements from a set of n elements.

Counting Subsets

k	k-element subsets of $\{a,b,c,d\}$	$\binom{4}{k}$
-1		$\binom{4}{-1} = 0$
0	$\emptyset$	$\binom{4}{0} = 1$
1	$\{a\},\{b\},\{c\},\{d\}$	$\binom{4}{1} = 4$
2	$\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\}$	$\binom{4}{2} = 6$
3	$\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}$	$\binom{4}{3} = 4$
4	$\{a,b,c,d\}$	$\binom{4}{4} = 1$
5		$\binom{4}{5} = 0$
6		$\binom{4}{6} = 0$

Example

- How many 5-card hands are there in which two of the cards are clubs and three are hearts?
- Ans: Note that order is not important.

How many ways can we select two club cards?

How many ways can we select three hearts?

Example

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How many ways can we select two club cards?

The answer is: C(13,2) = $\binom{13}{2}$ = $\frac{13!}{2!11!}$

How many ways can we select three hearts?

The answer is: C(13,3)

Total number = C(13,2). C(13,3) = $\frac{13!}{2!11!}\frac{13!}{3!10!}$

Example

- 3.3(4) Suppose a set B has the property that $|\{X : X \in P(B), |X| = 6\}| = 28.$ Find |B|.

- **Ans:**

Example

- 3.3(4) Suppose a set B has the property that $|\{X : X \in P(B), |X| = 6\}| = 28.$ Find |B|.

- **Ans: Since**

$$28 = |\{X : X \in P(B), |X| = 6\}| = \binom{|B|}{6},$$

therefore

$$\binom{|B|}{6} = \frac{|B|!}{(|B|-6)!6!} = 28.$$

- **The above identity is valid when |B|=8.**

Example

- In how many ways a gambler draw five cards from a standard deck and get
  - (a) a flush (five card of the same suit)?
  - (b) four aces?
  - (c) four of a kind?
  - (d) three aces and two jacks?
  - (e) three aces and a pair?
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  - (e) three aces and a pair? C(4,3).C(12,1).C(4,2)
  - (f) a full house (three of a kind and a pair)?
  - (g) three of a kind?
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  - (h) two pairs? C(13,2).C(4,2).C(4,2).C(44,1)

Example

- 3.3(8) The problem concerns lists made from the symbols A,B,C,D,E,F,G,H,I.
  - (a) How many length-5 lists can be made if repetition is not allowed and the list is in alphabetical order? (Example: BDEFI or ABCGH, but not BACGH)
    - Observe that size-5 subset realizes one length-5 list in alphabetical order, and one length-5 list realizes one size-5 subset. Therefore, the number od length-5 alphabetically ordered list is the same as the number of size-5 subsets. Hence the number is C(9,5).
  - (b) How many length-5 lists can be made if repetition is not allowed and the list is **not** alphabetical order.
    - Ans: 9x8x7x6x5 – C(9,5).

Example

- **3.3(14)**

Suppose $n, k \in \mathbb{Z}$, and $0 \leq k \leq n$. Use the definition alone (combinatorial argument) to show that $\binom{n}{k} = \binom{n}{n-k}$.

Ans: $C(n,k)$ is the number of ways of choosing size-k subsets of a set A of n objects. Note that by selecting a size-k subset, we also implicitly select a a size-(n-k) subset (which are left in A). Therefore $\binom{n}{k} = \binom{n}{n-k}$.

Pascal's Triangle and the Binomial Theorem

- We will focus on the pattern based on one equation (identity)

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

for any integers 1 ≤ n ≤ k.

- For any integers 1 ≤ n ≤ k,

C(n+1,k) = C(n,k-1) + C(n,k)

Pascal's Triangle and the binomial Theorem

- For any integers $1 \le k \le n$,

$$C(n+1,k) = C(n,k-1) + C(n,k)$$

- It says that number of size-k subset of a set with n+1 elements is equal to the sum of the number of size-(k-1) subset of a set of n elements and the number of size-k subset of a set of n elements.
- Consider a set with n+1 elements.

  A={0,1,2, …,n).
- All subsets of A of size k

  = All subsets of size k with element 0 in it

  + All subsets of size k with no element 0.

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Pascal's Triangle and the Binomial Theorem

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- Consider a set with n+1 elements

  A={0,1,2, …,n).

Note that k is at least 1 and k cannot be greater than n.

- All subsets of A of size k

  = All subsets of size k with element 0 in it

  + All subsets of size k with no element 0.

C(n+1,k) = C(n,k-1) + C(n,k).

Pascal's Triangle

- Suppose we write C(n,k), 1 ≤ k ≤ n for n=0,1,2, … and for all feasible values of k, with a certain pattern as follows:

n=0 -- $\binom{0}{0}$

n=1 ---------------------------------- $\binom{1}{0} \quad \binom{1}{1}$

n=2 ----------------------------- $\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$

n=3 ------------------------- $\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$

n=4 -------------------- $\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$

n=5 ---------------- $\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$

n=6 ------------ $\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$

n=7 -------- $\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$

$\cdots \quad \vdots \quad \cdots$

Pascal’s Triangle

- Any number C(n+1,k) for $1 \leq k \leq n$ is immediately below and between the two numbers C(n,k-1) and C(n,k) in the previous row.

Pascal's Triangle

- When we evaluate C(n,k) for all n and k, we get the following triangle of numbers. For example C(6,4)=20.
- The arrangement is called **Pascal's triangle (Pascal 1623-1660)**.

n=0 (Row 0)------------------------------	1
n=1 (Row 1)--------------------------	1 1
n=2 (Row 2) ---------------------	1 2 1
n=3 (Row 3)---------------	1 3 3 1
n=4 (Row 4)------------	1 4 6 4 1
n=5 (Row 5)-------	1 5 10 10 5 1
n=6 (Row 6)---	1 6 15 20 15 6 1
n=7 (Row 7)	1 7 21 35 35 21 7 1

⋰ ⋮ ⋱

Pascal's Triangle and Coefficients of $(x+y)^n$

- n^{th} row of Pascal'striangle lists the coefficients of $(x+y)^n$.

					1						
				1		1					
			1		2		1				
		1		3		3		1			
	1		4		6		4		1		
1		5		10		10		5		1	

$$1$$

$$1x + 1y$$

$$1x^2 + 2xy + 1y^2$$

$$1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

$$1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$

Binomial Theorem

Theorem 3.1 (Binomial Theorem) If n is a non-negative integer, then $(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$.

1	1
1 1	$1x + 1y$
1 2 1	$1x^2 + 2xy + 1y^2$
1 3 3 1	$1x^3 + 3x^2y + 3xy^2 + 1y^3$
1 4 6 4 1	$1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$
1 5 10 10 5 1	$1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$

Binomial Theorem

Want to show that

$$\mathbf{(x+y)^5 = \binom{5}{5}x^5y^0 + \binom{5}{4}x^4y + \binom{5}{3}x^3y^2 + \binom{5}{2}x^2y^3 + \binom{5}{1}xy^4 + \binom{5}{0}x^0y^5.}$$

We can write

$$(x+y)^5 = (x+y) \times (x+y) \times (x+y) \times (x+y) \times (x+y)$$

involving 5 factors.

Suppose we are interested in computing the coefficient x^3y^2.

Binomial Theorem

Want to show that

$$(\mathbf{x}+\mathbf{y})^5 = \binom{5}{5}\mathbf{x}^5\mathbf{y}^0 + \binom{5}{4}\mathbf{x}^4\mathbf{y} + \binom{5}{3}\mathbf{x}^3\mathbf{y}^2 + \binom{5}{2}\mathbf{x}^2\mathbf{y}^3 + \binom{5}{1}\mathbf{x}\mathbf{y}^4 + \binom{5}{0}\mathbf{x}^0\mathbf{y}^5.$$

We can write

$$(x+y)^5 = (x+y)\times(x+y)\times(x+y)\times(x+y)\times(x+y)$$

involving 5 factors.

Suppose we are interested in computing the coefficient x^3y^2.
Out of 5 factors, we select x from 3 factors, and the remaining two factors contribute two y's. There are $\binom{5}{3}$ ways to select 3 factors from the 5 factors. Once the factors involving x is known, there is only one way to chose the remaining two factors for y.

Therefore the coefficient of x^3y^2 is $\binom{5}{3}$.

Binomial Theorem (Example)

- **3.4(4)** Use the binomial theorem to find the coefficient of x^6y^3 in $(3x-2y)^9$.

Binomial Theorem (Example)

- **3.4(4)** Use the binomial theorem to find the coefficient of x^6y^3 in $(3x-2y)^9$.

  We can write $(3x - 2y)^9$ as $(a + b)^9$ where a = 3x and b = -2y. Now the coefficient of a^6b^3 is C(9,6).

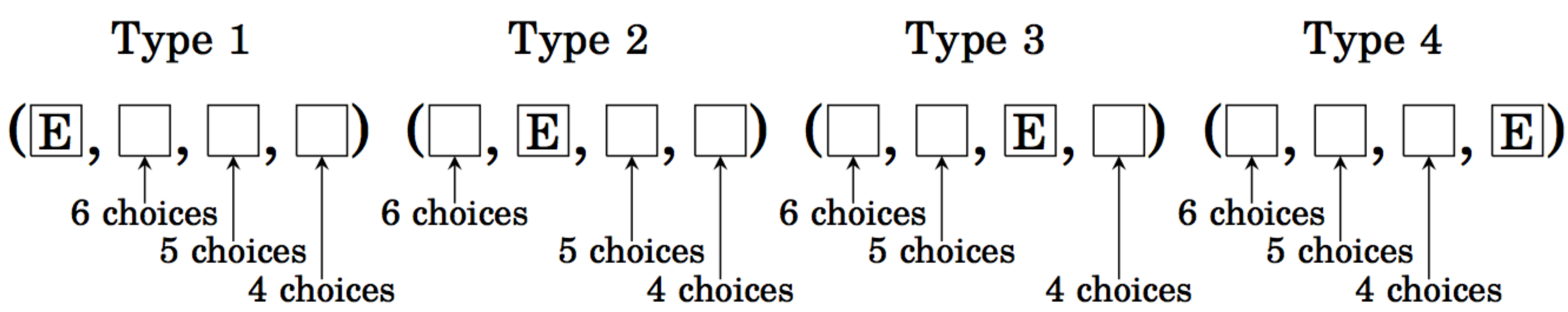
  Now $C(9,6).a^6b^3 = C(9,6)\ (3x)^6\ (-2y)^3 = -\ C(9,6)3^6\ 2^3\ x^6y^3$.

Inclusion-Exclusion (Section 3.5)

- We have looked at the following two problems.
- Consider making lists from the symbols A, B, C, D, E, F, G

  (c) How many length-4 lists are possible if repetition is **not** allowed and the list must contain an E?

  Ans: There are four types of lists depending on whether E occurs as the first, second, third or fourth entry. These four types are shown below.



Total # =(6 x 5 x 4) + (6 x 5 x 4) + (6 x 5 x 4) + (6 x 5 x 4)

$$|\cup_{i=1}^{i=4} Type-i-list| = \cup_{i=1}^{i=4}|Type-i-list|.$$

Inclusion-Exclusion (Section 3.5)

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2. Consider making lists from the symbols A, B, C, D, E, F, G

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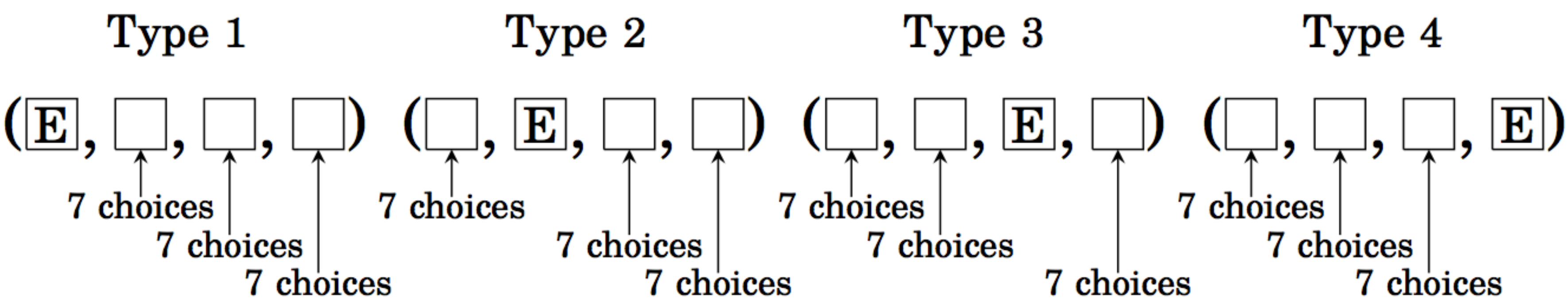
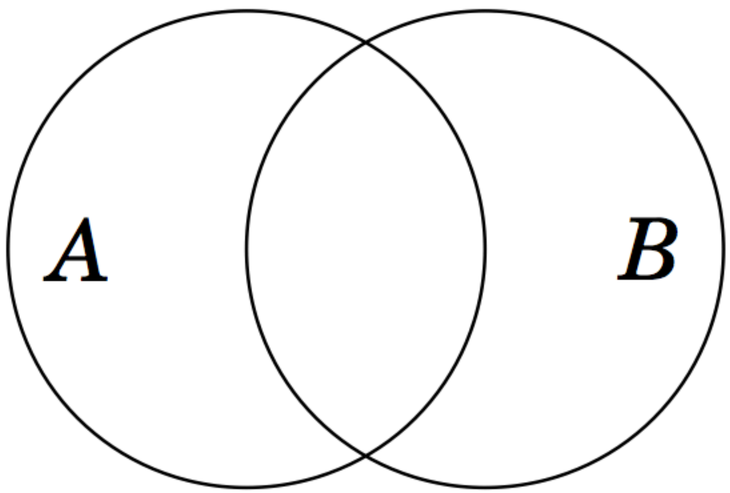


Total # =(7x7x7) + (7x7x7) + (7x7x7) + (7x7x7)

$$|\cup_{i=1}^{i=4} Type-i-list| \neq \sum_{i=1}^{i=4} |Type-i-list|.$$

Principle of Inclusion-Exclusion (PIE)

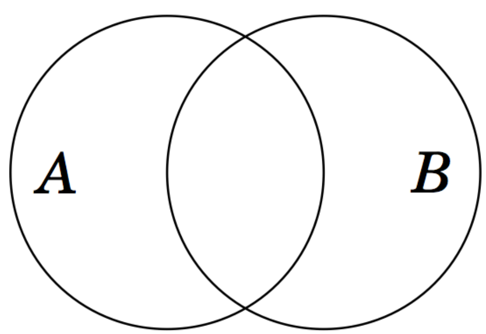
- It is not correct to say that $|A \cup B|$ must equal to $|A| + |B|$ if A and B have some elements in common. In this case we have counted each element of $A \cap B$ twice.



- Correct identity is $|A \cup B| = |A| + |B| - |A \cap B|$

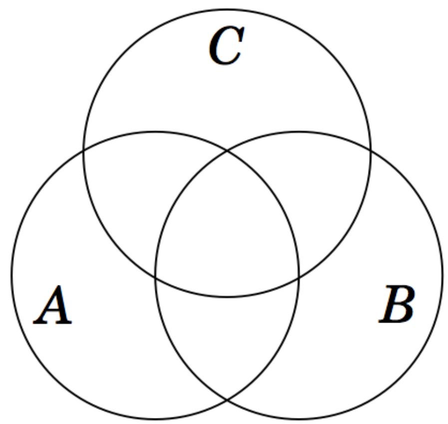
PIE

- More generally, we have the following
- **Lemma**: Let A, B, be subsets of a finite set U. Then
  1. $|A \cup B| = |A| + |B| - |A \cap B|$
  2. $|A \cap B| \leq \min\{|A|, |B|\}$
  3. $|A - B| = |A| - |A \cap B|$
  4. $|A \oplus B| = |A \cup B| - |A \cap B| = |A| + |B| - 2|A \cap B|$
  5. $|A \times B| = |A| \times |B|$



PIE

- Consider three sets A, B, C as represented in the following Venn Diagram.



- Using similar arguments as before we can write

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

PIE : Example

- To illustrate, when n=4, we have

$$
\begin{aligned}
|A_1 \cup A_2 \cup A_3 \cup A_4| = {} & |A_1| + |A_2| + |A_3| + |A_4| \\
& - [|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| \\
& \quad + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|] \\
& + [|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| \\
& \quad + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|] \\
& - |A_1 \cap A_2 \cap A_3 \cap A_4|
\end{aligned}
$$

PIE: Theorem

- **Theorem**: Let $A_1, A_2, \ldots, A_n$ be finite sets, then

$$
\begin{aligned}
|A_1 \cup A_2 \cup \ldots \cup A_n| = & \textstyle\sum_i |A_i| \\
& - \textstyle\sum_{i<j} |A_i \cap A_j| \\
& + \textstyle\sum_{i<j<k} |A_i \cap A_j \cap A_k| \\
& - \ldots \\
& + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|
\end{aligned}
$$

Each summation is over

- all i,
- pairs i,j with i<j,
- triples with i<j<k, etc.

Application of PIE: Example 1

- How many integers between 1 and 300 (inclusive) are
  - Divisible by at least one of 3,5,7?
  - Divisible by 3 and by 5 but not by 7?
  - Divisible by 5 but by neither 3 or 7?

Application of PIE: Example 1

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- Let

  $A = \{n \in Z \mid (1 \leq n \leq 300) \wedge (mod(n,3) = 0)\}$

  $B = \{n \in Z \mid (1 \leq n \leq 300) \wedge (mod(n,5) = 0)\}$

  $C = \{n \in Z \mid (1 \leq n \leq 300) \wedge (mod(n,7) = 0)\}$

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- How big are these sets? We use the floor function

  $|A| = \lfloor 300/3 \rfloor = 100$

  $|B| = \lfloor 300/5 \rfloor = 60$

  $|C| = \lfloor 300/7 \rfloor = 42$

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$\lfloor 16.3 \rfloor = 16$; $\lfloor 42.857142.... \rfloor = 42$

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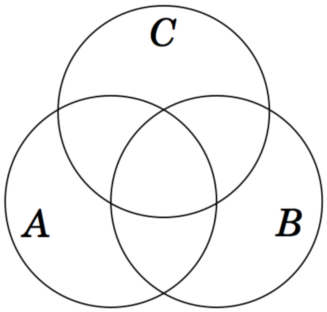
- How many integers between 1 and 300 (inclusive) are divisible by at least one of 3,5,7?

  Answer:



How big are these sets? We use the floor function

$|A| = \lfloor 300/3 \rfloor = 100$

$|B| = \lfloor 300/5 \rfloor = 60$

$|C| = \lfloor 300/7 \rfloor = 42$

Application of PIE: Example 1

- How many integers between 1 and 300 (inclusive) are divisible by at least one of 3,5,7?

  Answer: $|A \cup B \cup C|$

Application of PIE: Example 1

- How many integers between 1 and 300 (inclusive) are divisible by at least one of 3,5,7?

  Answer: $|A \cup B \cup C|$
- By the principle of inclusion-exclusion

  $$|A \cup B \cup C| = |A|+|B|+|C|-[|A \cap B|+|A \cap C|+|B \cap C|]+|A \cap B \cap C|$$
- How big are these sets? We use the floor function

  $|A| = \lfloor 300/3 \rfloor = 100$ $\quad$ $|A \cap B| = \lfloor 300/15 \rfloor = 20$

  $|B| = \lfloor 300/5 \rfloor = 60$ $\quad$ $|A \cap C| = \lfloor 300/21 \rfloor = 14$

  $|C| = \lfloor 300/7 \rfloor = 42$ $\quad$ $|B \cap C| = \lfloor 300/35 \rfloor = 8$

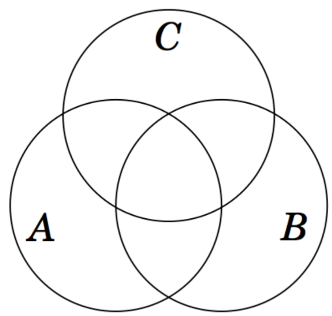
  $|A \cap B \cap C| = \lfloor 300/105 \rfloor = 2$
- Therefore:

  $$|A \cup B \cup C| = 100 + 60 + 42 - (20+14+8) + 2 = 162$$

Application of PIE: Example 1

- How many integers between 1 and 300 (inclusive) are divisible by 3 and by 5 but not by 7?

  Answer:



- Knowing that

$|A| = \lfloor 300/3 \rfloor = 100$

$|B| = \lfloor 300/5 \rfloor = 60$

$|C| = \lfloor 300/7 \rfloor = 42$

$|A \cap B| = \lfloor 300/15 \rfloor = 20$

$|A \cap C| = \lfloor 300/21 \rfloor = 100$

$|B \cap C| = \lfloor 300/35 \rfloor = 8$

$|A \cap B \cap C| = \lfloor 300/105 \rfloor = 2$

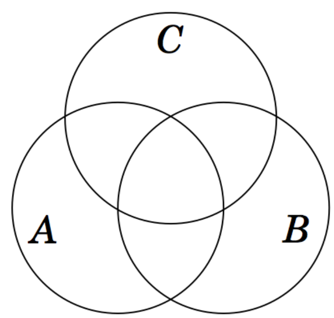
Application of PIE: Example 1

- How many integers between 1 and 300 (inclusive) are divisible by 3 and by 5 but not by 7?

  Answer: $|(A \cap B) - C|$

- By the definition of set-minus

$$|(A \cap B) - C| = |A \cap B| - |A \cap B \cap C| = 20 - 2 = 18$$



- Knowing that

  $|A| = \lfloor 300/3 \rfloor = 100$ $\quad$ $|A \cap B| = \lfloor 300/15 \rfloor = 20$

  $|B| = \lfloor 300/5 \rfloor = 60$ $\quad$ $|A \cap C| = \lfloor 300/21 \rfloor = 100$

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Application of PIE: Example 1

- How many integers between 1 and 300 (inclusive) are divisible by 5 but by neither 3 or 7?

  Answer:

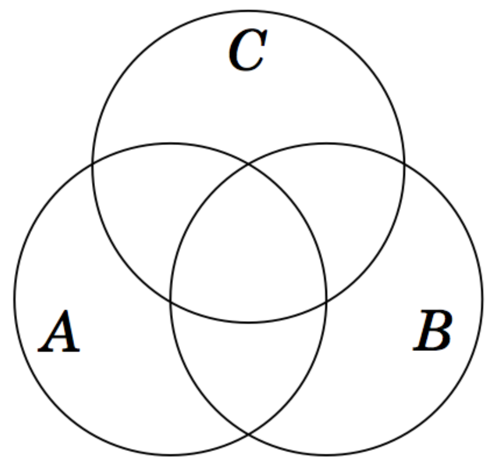
- Knowing that

  $|A| = \lfloor 300/3 \rfloor = 100$ $\quad$ $|A \cap B| = \lfloor 300/15 \rfloor = 20$

  $|B| = \lfloor 300/5 \rfloor = 60$ $\quad$ $|A \cap C| = \lfloor 300/21 \rfloor = 14$

  $|C| = \lfloor 300/7 \rfloor = 42$ $\quad$ $|B \cap C| = \lfloor 300/35 \rfloor = 8$

  $|A \cap B \cap C| = \lfloor 300/105 \rfloor = 2$



Application of PIE: Example 1

- How many integers between 1 and 300 (inclusive) are divisible by 5 but by neither 3 or 7?

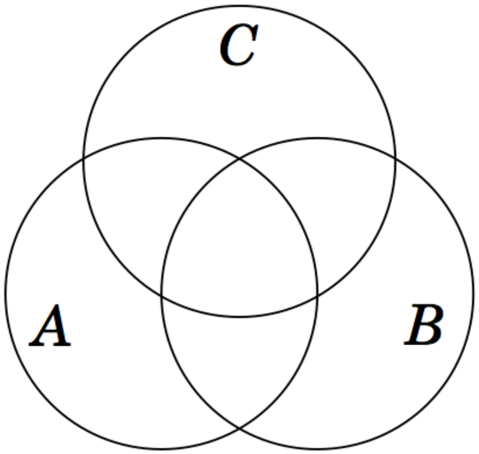
  Answer: $|B - (A \cup C)| = |B| - |B \cap (A \cup C)|$
- Distributing B over the intersection

  $|B \cap (A \cup C)| = |(B \cap A) \cup (B \cap C)|$

  $= |B \cap A| + |B \cap C| - |(B \cap A) \cap (B \cap C)|$

  $= |B \cap A| + |B \cap C| - |B \cap A \cap C|$

  $= 20 + 8 – 2 = 26$



- Knowing that

  $|A| = \lfloor 300/3 \rfloor = 100$ $\quad |A \cap B| = \lfloor 300/15 \rfloor = 20$

  $|B| = \lfloor 300/5 \rfloor = 60$ $\quad |A \cap C| = \lfloor 300/21 \rfloor = 14$

  $|C| = \lfloor 300/7 \rfloor = 42$ $\quad |B \cap C| = \lfloor 300/35 \rfloor = 8$

  $|A \cap B \cap C| = \lfloor 300/105 \rfloor = 2$

Solution

- 3.2(6) There are two 0's at the end of 10!=3,268,800. Using only paper and pencil, determine how many 0's are at the end od 100!.

  Ans: The two zeros in 10! come from the fact that
  - (1) The number is divisible by 10, and
  - (2) The number is divisible by both 5 and 2.

  Now 100! is divisible by 100*(95*92)* 90*(85*82)*
.....10*(5*2).
  - The number of trailing zeros in 100! is 21.

Solution

- 3.2(6) There are two 0's at the end of 10!=3,268,800. Using only paper and pencil, determine how many 0's are at the end od 100!.

  Ans: The two zeros in 10! come from the fact that

  - (1) The number is divisible by 10, and
  - (2) The number is divisible by both 5 and 2.

  The number of trailing zeros in 100! is ~~21~~ **24**

Solution

- 3.2(6) There are two 0's at the end of 10!=3,268,800. Using only paper and pencil, determine how many 0's are at the end od 100!.

Ans: The number of trailing zeros in 100! is ~~21~~ **24**

Number	Zeros	Number	Zeros
100	2	95	1
90	1	85	1
80	1	75	2
70	1	65	1
60	1	55	1
50	2	45	1
40	1	35	1
30	1	25	2
20	1	15	1
10	1	5	1
Total	**12**	**Total**	**12**

'**zeros**' columns indicate the number of factors of 5.

Solution

- 3.2(6) There are two 0's at the end of 10!=3,268,8 $\frac{100!}{95!}$.
Using only paper and pencil, determine how many 0's are at the end od 100!.

Ans: The two zeros in 10! come from the fact that

One of the students points out that there are 24 trailing zeros. I have counted two factors of 5 in 100, but didn't count two factors of 5 in 75, 50 and 25. These extra three factors of 5 realize another 3 extra trailing zeros, when multiplied with a real number.

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Example 2:

- **3.5(2)** How many 4-digit positive integers are there for which there are no repeated digits, or for which there may be repeated digits, but all are odd.
- A = set of length-4 lists of integers with no repeated digits.
- B_i = set of length-4 lists ending with odd digit i, repetition is allowed, i=1, 3, 5, 7, 9
- C_i = set of length-4 lists ending with odd digit i, repetition is not allowed, i=1, 3, 5, 7, 9
- $B_i – C_i$ is the set of length-4 odd numbered list where at least one digit is repeated.
- **Ans to the question is:**

Example 2:

- **3.5(2)** How many 4-digit positive integers are there for which there are no repeated digits, or for which there may be repeated digits, but all are odd.
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- C_i = set of length-4 lists ending with odd digit i, repetition is not allowed, i=1, 3, 5, 7, 9
- $B_i – C_i$ is the set of length-4 odd numbered list where at least one digit is repeated.
- **Ans to the question is:**

$$|A| + |B_1 - C_1| + |B_3 - C_3| + |B_5 - C_5| + |B_7 - C_7| + |B_9 - C_9| .$$

Practice problems from the text:

- Section 3.1
  - 1, 2, 3, 4, 5, 8, 9, 12
- Section 3.2
  - 3, 5, 7
- Section 3.3
  - 1, 3, 5, 7, 9, 10, 11, 12
- Section 3.4
  - 3, 5, 6, 8, 9, 11, 13
- Section 3.5
  - 1, 3, 4, 5, 6, 8