1 Permutations

Suppose S is a set of n distinct objects. A **permutation** of S is an ordered arrangements of these objects. This is called a list in the text. Thus, for any k, $0 \leq k \leq n$, a k-permutation is a length-k list. We denote by $P(n,k)$ to be the number of k-permutations of a set of n distinct objects. Thus

$$P(n,k) = \frac{n!}{(n-k)!}$$

.

1.1 Permutations with repetitions

The number of k-permutations (length-k lists) of a set of n objects with repetition allowed is n^k.

This has been discussed in the class. This is nothing but generating lists with repetitions.

1.2 Permutations with indistinguishable objects

This is not directly discussed in Chapter 3 of the text. So far we have assumed that the objects are distinct. We will now consider the case when some elements may be indistinguishable (not distinct).

Example How many different lists one can make by reordering the letters of the word $JESSEE$? (Note that '$JESSEE$' and '$SJESEE$' are two different words (lists).) We discuss two methods.

First method

We first indicate three Es as E^1, E^2, E^3 and two Ss as S^1, S^2. Thus '$JESSEE$' can be written as '$JE^1S^1S^2E^2E^3$'. If all the three Es and two Ss are treated as distinct, we know that the number of permutations of length 6 is 6!. Now note that '$JE^1S^1S^2E^2E^3$' and '$JE^1S^2S^1E^2E^3$' are the same. Therefore, the number lists using the letters of '$JE^1SSE^2E^3$' is $\frac{6!}{2!}$. Similarly, in '$JE^1SSE^2E^3$' reordering the 'E^1, E^2, E^3' will result in the same list, the number of lists using the letters of '$JESSEE$' is $\frac{6!}{3!2!}$.

Second Method

The word '$JESSEE$' contains 3 Es, 2Ss and one J. We note that 3Es can be placed among the six positions in $C(6,3)$ different ways. There are three more positions to be filled once Es are placed. Two of these positions will be occupied by two Ss. There are $C(3,2)$ ways to fill in these positions. We are left with just one position, after 3 Es and 2Ss are placed, which is occupied by J. The number of ways to place J is $C(1,1)$. Using the product rule we can conclude that the number of different odds using the letters of '$JESSEE$' is $C(6,3).C(3,2).C(1,1) = \frac{6!}{3!3!} \times \frac{3!}{2!1!} \times \frac{1!}{1!0!} = \frac{6!}{3!2!1!}$.

Theorem: The number of different permutations of n objects, where there are n_1 type 1 similar (indistinguishable) objects, n_2 type 2 similar objects, ..., and n_t type t similar (indistinguishable) object, is $\frac{n!}{n_1!n_2!.....n_t!}$.

Distributing Presents (Another version of the above theorem)

We have n distinct gifts to be distributed to $k, k \leq n$, children such that child i gets exactly n_i gifts. We assume $\sum_{i=1}^{t} n_i = n$. The number of ways to distribute n gifts to k children is $\frac{n!}{n_1!n_2!.....n_t!}$.

Note here that the order of the gifts child i receives is not important.

2 Combinations

A **k-combination** of elements of a set is an unordered selection of k elements from the set. When repetitions are not allowed, a k-combination is a size-k subset of the set. Thus, for any $k, 0 \leq k \leq n$, the number of k-combination of a set of n elements is

$$C(n,k) = \frac{n!}{k!(n-k)!}.$$

Note the identities $C(n,k) = \frac{P(n,k)}{k!}$, and $C(n,k) = C(n,n-k)$. The Pascal triangle identity $C(n+1,k) = C(n,k-1) + C(n,k)$ is very well known, and important. We can prove these identities analytically by applying the properties of factorial. We should also be able to prove these identities using combinatorial arguments. A **combinatorial proof** of an identity is a proof that uses the counting arguments to prove that both sides of the identity realize the same number.

2.1 Combinations with repetitions

This topic is not covered in the text.

Consider the following problem, called **distribution of money**.

We have n pennies that we want to distribute to k kids. Each child gets at least one penny. How many ways can we distribute the money?

Note that the pennies are not distinct. There will be only one way of distributing n_1 pennies to the first kid, n_2 pennies to the second kid, and so on where $n_1 + n_2 + ... + n_k = n$.

Let us consider the following experiment.

- Line up the pennies, order doesn't matter; they are all alike
  x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x

- Let the first child pick them from left to right. After a while we stop the kid.
  x x x x x x x x x x x x x **stop** x x x x x x x x x x x x x x x x x
  kid 1 gets these coins

- Let the second child picks up pennies, etc.
  x x x x x x x x x x x x x **stop** x x x x x x **stop** x x x x x x x x x x x

  There are three kids. The first kid gets 13 pennies, the second kid gets 6 pennies and the last one gets the remaining 11 pennies.

The distribution of money is determined by specifying where to start with a new child. The first child always start from position 1. The other $k-1$ kids can enter at position 2, 3, 4, ..., n. This means that there are $C(n-1, k-1) = \binom{n-1}{k-1}$ ways to choose the entry points. Therefore,

Theorem: There are $C(n-1, k-1)$ ways to distribute n pennies to k kids with the condition that each kid gets at least one penny.

Now we would like to relax the condition that each kid gets at least one penny. Now it is possible that some kid doesn't receive any penny.

We use the following trick. We borrow one penny from each kid, and then distribute $(n+k)$ pennies to k kids such that each kid gets at least one penny.

Theorem: There are $\binom{n+k-1}{k-1}$ ways to distribute n pennies to k kids with no restrictions. This is the same as selecting k objects with repetitions where order is not important.

The above problem can be formulated differently in the following equivalent one. Let $x_i, i = 1, 2, ..., k$ be integer variables. The number of integer solutions to the equation

$$x_1 + x_2 + ... + x_k = n, \ \ x_i \geq 0 \text{ ———————(A)}$$

is the same as the number of ways to distribute n pennies to k kids with no restrictions. Consider a solution $(\alpha_1, \alpha_2, ..., \alpha_k)$ where $\sum_{i=1}^{k} \alpha_i = n$. Moreover $\alpha_i \geq 0 \ \forall i$. This solution corresponds to a solution of distributing n pennies to k kids, with no restrictions, where kid i gets α_i pennies. Similarly, a solution of distributing n pennies to k kids with no restrictions corresponds to a solution to the integral equation where the value of x_i is the number of gifts kid i gets. Thus we can conclude that the number of integral solutions to equation (A) is $C(n+k-1, k-1)$ which is the same as the number of distributing n pennies to k kids with no restrictions.

What happens when we place the restriction that each kid must receive at least one penny? This is equivalent to the problem of determining the number of integral solutions to

$$x_1 + x_2 + ... + x_k = n, \ \ \mathbf{x_i \geq 1} \text{ ———————(B)}$$

In some literature the example of distributing pennies to the kids is replaced by the example of distributing balls (pennies) to the bins (kids). In this case the bins are distinct (distinguishable) and balls are similar (indistinguishable).

How do we tie equation (A) with k-combinations of n distinct objects. In this case, x_i denotes the i^{th} distinct objects (bins), and k represents the number of indistinguishable (unordered) objects (balls) to be selected. Placing a ball in bin x_i indicates that i^{th} object is selected.

Example: Consider the following programming piece

```
for i= 1 to 100 do
   for j= 1 to i do
      for k= 1 to j do
         print(i*j+k)
```

How many times the print statement is executed?

Ans: The number of times the print statement is executed is

$$\sum_{i=1}^{100} \sum_{j=1}^{i} \sum_{k=1}^{j} 1 = \sum_{i=1}^{100} \sum_{j=1}^{i} j = \sum_{i=1}^{100} \frac{i(i+1)}{2} = ??.$$

There is another way to count this. This count can be converted into placing three indistinguishable balls into 100 distinguishable bins without any constraints. Let a, b and c be the bins where three balls are placed, $a \leq b \leq c$. Note that bins need not be distinct. Since $1 \leq k \leq j \leq i \leq 100$, we assign $k = a, \ j = b$ and $i = c$.

We can also see that any i, j and k in the code is also a solution to distributing 3 balls to 100 bins. Thus the number of times the print statement will be executed is exactly the same as placing three balls into 100 bins without restrictions, which is $\binom{3+100-1}{100-1} = \binom{102}{99} = \binom{102}{3} = \frac{102.101.100}{3.2.1}$ times.

We can now summarize permutations and combinations without and with repetition in the following table.

Type	Repetition Allowed?	Formula
k-permutations	No	$P(n,k) = \frac{n!}{(n-k)!}$
k-permutations	Yes	n^k
k-combinations	No	$C(n,k) = \frac{n!}{k!(n-k)!}$
k-combinations	Yes	$C(n+k-1, n-1) = \frac{(n+k-1)!}{(n-1)!k!}$

Example: Consider the set $\{a,b,c,d\}$. Suppose we select two letter from these four. We now have the following four cases.

A. Permutations with repetitions: In this case there are $4^2 = 16$ possible cases.

aa	ab	ac	ad
ba	bb	bc	bd
ca	cb	cc	cd
da	db	dc	dd

B. Permutations without repetitions: In this there are $P(4,2) = 12$ possible permutations.

	ab	ac	ad
ba		bc	bd
ca	cb		cd
da	db	dc	

C. Combinations with repetitions: There are $C(2+4-1, 4-1) = 10$ possible combinations.

aa	ab	ac	ad
	bb	bc	bd
		cc	cd
			dd

D. Combinations without repetitions: There are $C(4,2) = 6$ possible combinations.

	ab	*ac*	*ad*
		bc	*bd*
			cd

Test bank of problems

1. ***Find the number of bit strings of length 100

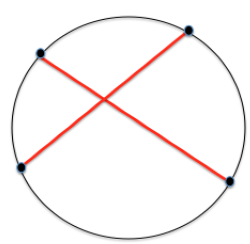
   (a) that begin with 1 and end with 0.

   (b) that begin with 1 or end with 0.

   (c) that have exactly 20 locations with 1s.

   (d) that have exactly 20 locations with 1s and none of these 1s are adjacent to each other.

2. ***Ten points are placed on the circumference (boundary) of a circle, and all the chords connecting these points are drawn. What is the largest number of intersection of these chords?
   { Any four points determine a pair of chords that intersect.}



3. ***How many ways can the 11 distinct horses be lined up in a row?

4. ***How many ways can the 11 distinct horses be lined up in a cycle? Note that (in case of 3 horses) $< a,b,c >$ order is the same as $< b,c,a >$ and $< c,a,b >$. In this case, there is no position 1, position 2 or position 3. The correct answer is $\frac{11!}{11}$ which is 10!.

5. ***How many ways can the 11 identical horses on a cycle be painted so that three are brown, three are white and five are black?

6. ***Find the number of lists (permutations) of the word

   (a) the word *SASKATCHEWAN*

   (b) the word *BATTERED*

   (c) the word *COEFFICIENT*

   (d) the word *BHATTACHARYYA*

7. ***You have 50 books (25 computer science books, 15 mathematics books and 10 engineering books). All books are different. In how many ways can you:

   (a) put 50 books in a row on one shelf?

   (b) put 50 books in a row on three shelves, each shelf containing at least one book.
   $\{$**Hint:** Each permutation of 30 books can be split into three parts in $\binom{29}{2}$. $\}$

   (c) put 50 books in a row on three shelves such that one shelf contains CS books only, another self contains math books only, and the remaining self contains engineering books only.

   (d) get bunch (subset) of 7 CS books and 5 engineering books to a friend.

8. ***Make up a word problem in English whose answer is

   (a) $\binom{25}{7} \times \binom{10}{3}$.

   (b) 2^n.

   (c) $2^n - 2$.

9. ***Let $S = \{1,2,3,4,5,6,7,8,9,0\}$. Find the number of subsets of S that contain:

(a) both 4 and 8.

(b) neither 4 nor 8.

(c) either 3 or 4 or both.

(d) no odd numbers.

(e) exactly 4 elements, one of which is 0.

(f) exactly five elements including, the sum of which is even.

10. ***Let A be the set of all words of length 6 of letters of the alphabet with no repeated letters (counting lists).

    (a) How many elements of A has exactly one vowel?

    (b) How many elements of A that begins and ends with a vowel.

11. ***Use binomial theorem to prove the following:

    (a) $\sum_{i=0}^{10} \binom{10}{i} = 2^{10}$.

    (b) $\sum_{i=0}^{50} \binom{100}{2i} = \sum_{i=1}^{50} \binom{100}{2i-1}$.

12. ***Find:

    (a) the coefficient of x^9y^3 in the expansion of $(4x-2y)^{12}$.

    (b) the coefficient of x^5 in $(2+x^2)^9$.

    (c) the coefficient of $x^3y^4z^2$ in the expansion of $(x+y+z)^{10}$.

    (d) largest coefficient in the expansion of $(1+x)^8$.

    (e) largest coefficient in the expansion of $(1+x)^9$.

13. Determine the number of integer solutions to the following:

    (a) equation $x_1+x_2+x_3=6, x_i \geq 0, 1 \leq i \leq 3$.

    (b) equation $x_1+x_2+x_3+x_4+x_5=15, x_i \geq 0, 1 \leq i \leq 5$.

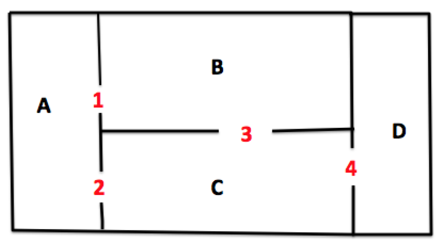
    (c) equations $x_1+x_2+x_3=6$ and $x_1+x_2+x_3+x_4+x_5=15, x_i \geq 0, \quad 1 \leq i \leq 5$.

    Answer: For $x_1+x_2+x_3=6$ there are $\binom{3+6-1}{3-1} = \binom{8}{2}$ nonnegative integer solutions. With $x_1+x_2+x_3=6$ and $x_1+x_2+x_3+x_4+x_5=15$, the number of nonnegative integer solutions for $x_4+x_5=9$ is $\binom{2+9-1}{2-1}$. The number of solutions for the pair of equations is $\binom{8}{6} \times \binom{10}{1}$.

(d) equations $x_1+x_2+x_3 \le 6$ and $x_1+x_2+x_3+x_4+x_5 \le 15, x_i \ge 0, \quad 1 \le i \le 5$.

Answer: Let $0 \le k \le 6$. For $x_1+x_2+x_3 = k$, there are $\binom{3+k-1}{3-1} = \binom{k+2}{2}$ solutions. To solve $x_4+x_5 \le 15-k$, consider $x_4+x_5+y = 15-k$, $x_4, x_5, y \ge 0$. There are $\binom{3+15-k-1}{3-1} = \binom{17-k}{2}$ solutions. The total number of solutions is $\sum_{k=0}^{6} \binom{k+2}{2} \times \binom{17-k}{2}$.

14. ***Four connecting rooms are to be painted with k district colors so that no two adjacent rooms have the same color. Room A is connected to room B (by door 1) and room C (by door 2). Room B is connected to room C (by door 3). Room C is connected to room D by door 4. How many ways can one paint the rooms so that no two adjacent rooms (shared by a door) cave the same colour.



{ Let D_i be the painting schemes if the wall containing door i is removed. $D_i \cap D_j$ is set of the painting schemes coloring the rooms if the walls containing i and j are removed. We can define similarly $D_i \cap D_j \cap D_k$ and $D_i \cap D_j \cap D_k \cap D_l$. Note that when all 4 walls containing doors are removed, there are only k ways to paint one room. If there is no constraint on colors of adjacent rooms, k^4 different ways we can paint four rooms. }

15. Some questions from a past midterm.

(a) (10 points) Consider selecting 4 objects from the set $A = \{1,2,3,4,5,6,7,8\}$.

i. How many ordered sequences without repetition can be chosen from A? **ans= P(8,4)**

ii. How many ordered sequences with repetition can be chosen from A? **ans=** 8^4**; there are 8 choices for each position.**

iii. How many unordered sequences without repetition can be chosen from A? **ans= C(8,4)**

iv. How many unordered sequences with repetition can be chosen from A? $\binom{8+4-1}{8-1}$**; this r-combinations with repetitions.**

v. How many strictly increasing sequences can be chosen from A? $\{<2,4,4,7>$ is not a strictly increasing sequence.$\}$
It is the same as the number of $4-$combinations without repetitions, since every such $4-$element combination, there is only one strictly increasing sequence. Hence the answer is C(8,4).

(b) (10 points) Consider a eight letter word *aeemrryt*.

i. How many different arrangements of these seven letters are there?
no constraint: $\frac{8!}{2!2!}$

ii. How many such arrangements are there that contain *eye*?
Arrangements with eye: use $\{eye, a, r, m, r, t\}$: $\frac{6!}{2!}$

iii. How many such arrangements are there that contain *eye* and *ram*?
Arrangements with eye and ram: use $\{ram, eye, r, t\}$: $4!$

iv. How many such arrangements are there that do not contain either *eye* or *ram*?
Arrangements with neither eye nor ram $= \frac{8!}{2!2!} - \frac{6!}{2!} - \frac{6!}{2!} + 4!$

(c) (10 points) Suppose you are interested in buying pizzas, and each pizza gets up to 10 distinct toppings.

i. How many ways can you choose toppings for a pizza?
There are 2^{10} **different pizzas.**

ii. How many ways can you choose two pizzas with the same toppings?
It is the same as the number of different pizzas. The answer is 2^{10}**.**

iii. How many ways can you choose toppings for two pizzas?
Since we can have two pizzas with the same toppings, the problem is combination with repetitions. There are 2^{10} **different pizzas, and we need to select two of them where repetitions are allowed. Therefore, the answer is** $\binom{2+2^{10}-1}{2^{10}-1} = \binom{2^{10}+1}{2}$

iv. How many ways can you choose toppings for n pizzas?
We now select n **pizzas from** 2^{10} **different toppings ones. The answer is** $\binom{n+2^{10}-1}{2^{10}-1}$**.**

(d) (10 points) We have seen that the following problem captures many counting problems.
Determine the number of non-negative integer solutions to

$$x_1 + x_2 + ... + x_k = n$$
$$x_i \geq 0, i = 1, 2, ..., k.$$

Formulate each of the following problems as a variation of the above problem.

i. Determine the number of ways to select k objects with replacements from a set of n objects.
Ans:

$$x_1 + x_2 + ... + x_n = k$$
$$x_i \geq 0, i = 1, 2, ..., n.$$

ii. Determine the number of ways to place n nondistinguisable balls in k boxes.
Ans:

$$x_1 + x_2 + ... + x_k = n$$
$$x_i \geq 0, i = 1, 2, ..., k.$$

iii. Determine the number of ways to distribute n pennies to k kids such that each kid gets at least 1 penny.
Ans:

$$x_1 + x_2 + ... + x_k = n$$
$$x_i \geq 1, i = 1, 2, ..., k.$$

iv. Determine the number of times the following pseudocode prints the PRINT statement:

```
for i = 1 to 20
   for j = i to 20
      for k = j to 20
         PRINT(i,j,k)
```

Ans:

$$x_1 + x_2 + ... + x_{20} = 3$$
$$x_i \geq 0, i = 1, 2, ..., 20.$$

Once the three integers are selected, we assign the largest one to k, the smallest one to i and the third one to j.