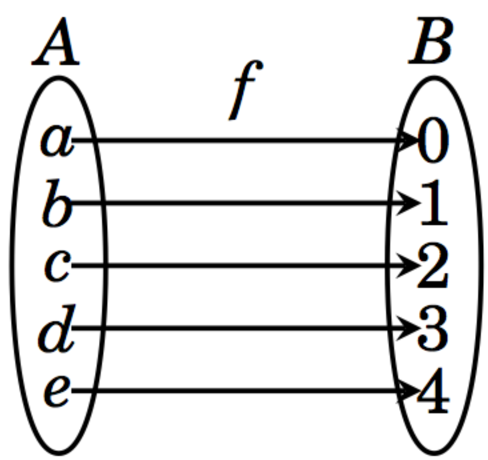
Cardinality of Sets

Sets with equal cardinalities

- Definition: Two sets A and B have the **same cardinality**, written |A| = |B|, if there exists a bijective function f: A → B. If no such bijective function exists, then the sets have **unequal cadinalities**, i.e. |A| ≠ |B|.



Finite set

- When A and B are finite, and |A| = |B|, it is easy to design a bijective function f: A $\rightarrow$ B.
  - Let A = $\{a_1, a_2, \ldots., a_k\}$ and B = $\{b_1, b_2, \ldots, b_k\}$.
  - Define f = $\{(a_1, b_1), (a_2,b_2), \ldots, (a_k, b_k)\}$.
  - f is bijective.

Infinite sets

- Are there more natural numbers N than there are integers Z?
- Let f: N $\rightarrow$ Z be a function defined by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ -\frac{(n-1)}{2} & \text{if } n \text{ is odd.} \end{cases}$$

- Now f is one to one and onto. (Why? Exercise)
- Some values of the function.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	$\ldots$
$f(n)$	0	1	-1	2	-2	3	-3	4	-4	5	-5	6	-6	7	-7	$\ldots$

- Therefore, |N| = |Z|.

Does |N| = |Q|?

- Note that $\mathbb{Q} = \{\frac{x}{y} | \, x, y \in \mathbb{Z}, \; y \neq 0\}$
- Another approach.
- Since N is a subset of Q, |N| ≤ |Q|.
- Consider the following one-to-one function f : Q → N.

$$f(n) = \begin{cases} 2^a.3^b & \text{if } n > 0 \text{ and } n = \frac{a}{b} \text{ in simplified form,} \\ 1 & \text{if } n = 0, \\ 2^a.3^b.5 & \text{if } n < 0 \text{ and } n = -\frac{a}{b} \text{ in simplified form.} \end{cases}$$

- Since f is one-to-one, |Q| ≤ |N|
- Therefore, |Q| = |N|

Countable and Uncountable Sets

- Definitions: Suppose A is a set. Then A is countably infinite if $|N|=|A|$, that is, there exists a bijection f: N $\rightarrow$ A. The set A is uncountable if A is infinite and $|N| \neq |A|$, that is, if A is infinite and there exists no bijection f: N $\rightarrow$ A.

Countable and Uncountable Sets

- Theorem: A set is countably infinite if and only if its elements can be arranged in an infinite list a_1, $a_{2,}$ $a_{3,}$ ….. Here f(i) = a_i.
- Prove that the set of natural numbers N= {1,2,3,4,…} has the same cardinality as the set E={2,4,6,8, …} of positive even integers.

n	1	2	3	4	5	6	7	8	9	10	11	12	13
f(n)	2	4	6	8	10	12	14	16	18	20	22	24	26

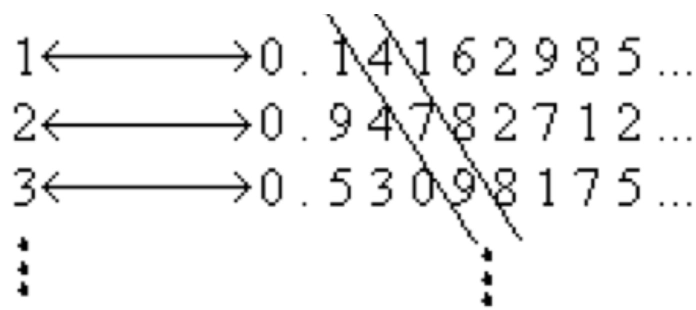
- Define f: N $\rightarrow$ Z by f(n) = 2n
- $f^{-1}(n) = n/2$

Does |N| = |R|?

- We have established that N, Z and Q all have the same cardinalities.
- Fact: between any two real numbers, there is always a rational number.
  - Surprising that there are more real numbers than rationals!
- We now show that even the real numbers in the interval [0,1] are not countable.
- Recall that a real number can be written out in an infinite decimal expansion.
- A real number in the interval [0,1] can be written as $0.d_1d_2d_3$....; 1 = 0.999999.......

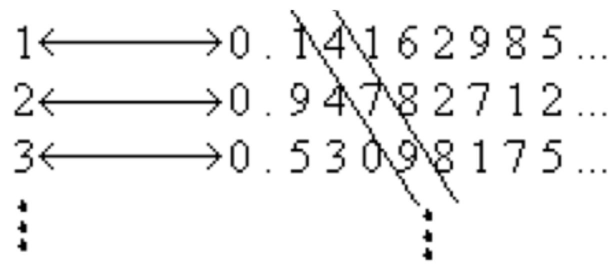
Showing that f: N → R(0,1) is not onto.

- Proof by contradiction.
- Suppose that f is onto. We can then enumerate the infinite list as follows:



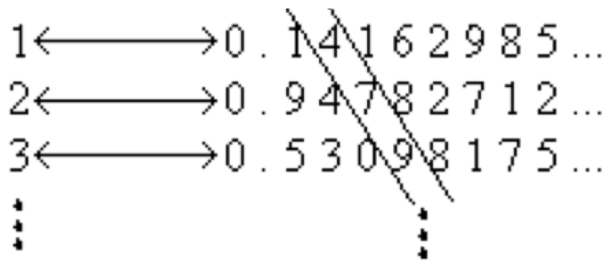
- The number circles in the diagonal is some real number r, since it is an infinite decimal expansion.
- Consider a real number s obtained by modifying every digit of r, say by replacing each digit d with d+5 mod 10.

Showing that f: N → R(0,1) is not onto.



- The number circles in the diagonal is some real number r, since it is an infinite decimal expansion.
- Consider a real number s obtained by modifying every digit of r, say by replacing each digit d with d+5 mod 10.
- We claim that s does not occur in our infinite list of real numbers. This is due to the fact that the n^{th} digit of s is different from the n^{th} digit of the n^{th} number from the list.
- Therefore, s is not in range(f).
- Therefore, f is not onto. Hence R is not countable.

Showing that f: N → R(0,1) is not onto.



Cantor's Diagonalization

- The number circles in the diagonal is some real number r, since it is an infinite decimal expansion.
- Consider a real number s obtained by modifying every digit of r, say by replacing each digit d with d+5 mod 10.
- We claim that s does not occur in our infinite list of real numbers. This is due to the fact that the n^{th} digit of s is different from the n^{th} digit of the n^{th} number from the list.
- Therefore, s is not in range(f).
- Therefore, f is not onto. Hence R is not countable.

Example

- $|(0, 1)| = |(1, \infty)|$
  - Let $f: (0,1) \to (1, \infty)$ be defined by $f(x) = 1/x$.
  - $f^{-1}(x) = 1/x$

Some results

- Any subset of a countable set is countable.
- If A and B are countable, A $\cup$ B is a countable set.
  - list the elements a_1, b_1, a_2, b_2, …
- Let X be an uncountable set. If there is a bijective function f : A → X, then A is uncountable.