Solution To Homework 8

1. **a)** $R_1 \cup R_2 =$
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$
b) $R_1 \cap R_2 =$
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$
c) $R_1 \times R_2 =$
$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
d) $R_1 \oplus R_2 =$
$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

2. **a)** All relations whose elements in the defined set are only related to themselves are both symmetric and anti-symmetric, e.g. $R = \{(a, a), (b, b), (c, c)\}$

b) Since R is a partial ordering, so R is anti-symmetric which means if aRb and bRa, then $a = b$. Besides $R = R^{-1}$ which means if aRb then bRa, it is actually symmetry. R is a relation both symmetric and anti-symmetric, same as (a).

c) Relations can be represented as a set of ordered pairs:
$R = \{(1, 1), (2, 2), (1, 2), (2, 1), (3, 3), (4, 4), (5, 5), (6, 6), (4, 5), (5, 4), (5, 6), (6, 5), (4, 6), (6, 4)\}$

d) Do not exist. Proof by contradiction:
Suppose x is minimal element and y is minimum($x \neq y$).
Since y is minimum: $\forall a \in A$, y is related to a. Therefore y is related to x.
And x is minimal: $\forall a \in A$, a is not related to x, so y cannot be related to x, which is a contradiction. (QED)

3. Relations with only one equivalence class {1,2,3}:
$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3), (1, 3), (3, 1)\}$
Two equivalence classes(take $\{1, 2\} \cup \{3\}$ as an example):
$R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$
Three equivalence classes $\{1\} \cup \{2\} \cup \{3\}$:
$R_3 = \{(1, 1), (2, 2), (3, 3)\}$

4. **a)** R is not an equivalence relation, because it is not reflexive. For reflexivity we need to have $(x, x) \in R$ for all $x \in \mathbb{R}$. But we have $1 = 1$, so $(1, 1) \notin R$.

b) S is not a partial order because it is not reflexive. For reflexivity we must have for all $x \in \mathbb{Z}$, $(x, x) \in S$. Since $0 \nmid 0$, $(0, 0) \notin S$. So S in not reflexive.

5. Suppose (A, R) is a poset. Proof by contradiction: Assume that there are (at least) two different greatest elements m_1 and m_2 ($m_1 \neq m_2$). Since m_1 and m_2 are greatest elements, we have $\forall a \in A \quad (a, m_1) \in R$ and $(a, m_2) \in R$. In particular we have $(m_2, m_1) \in R$ and $(m_1, m_2) \in R$. By antisymmetry we have $m_1 = m_2$. Contradiction.