Pigeonhole Principle includes solutions to PartB questions

Homework #7

Date due: April 2, 2014

The pigeonhole principle (PHP)

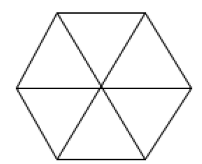
- If m pigeons occupy n pigeonholes, and $m > n$, then at least one pigeonhole has two or more pigeons roosting in it.
- The PHP is a powerful tool to solve combinatorial problems.
- The next slide discusses the main theorem and its proof. We will list a host of problems, categorized into two parts.
- Part A problems are for your practice.
- The homework questions are listed in Part B section.

Generalized pigeonhole principle

- If m objects are placed into n boxes, then there is at least one box containing $\lceil m/n \rceil$ objects
  - Proof by contradiction. Suppose each box contains less than $\lceil m/n \rceil$ objects, and there are m objects in total in n boxes. In this case the total number of objects n boxes can hold is at most $(\lceil m/n \rceil -1)*n$ which is less than $(((m/n)+1)-1)*n$, since $\lceil x \rceil < x+1$ always. This implies that there are less than m pigeons in n boxes. This is a contradiction.

PART A

1. Let $S \subset Z^+$, where $|S|=12$. Then S contains two elements that have the same remainder upon division by 11. (Discussed in the class)
2. Example 5.45 of the text is discussed in the class.
3. 19 darts are thrown onto a dartboard which is shaped as a regular hexagon with side length of 1 unit. Show that there are 4 darts within distance $(\sqrt{3})/3$.



4. Show that among 200 people, there are at least 17 people who are born on the same month
5. How many students in a class must there be to ensure that 10 students get the same grade (one of A, B, C, D, F, or N)?

PART A (contd.)

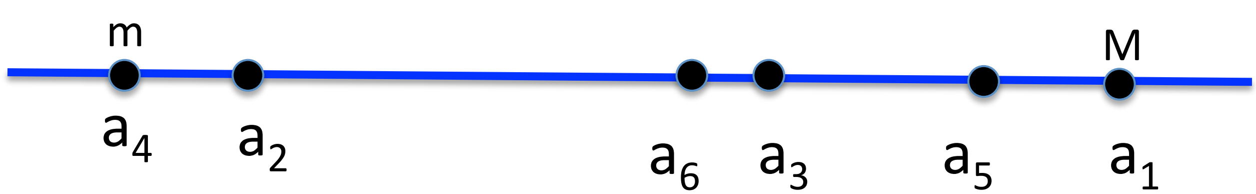
6. Suppose that there are 50 people in the room. Some of them are acquainted with each other, while some are not. Assume that each person has at least one acquaintance. Show that there are two persons in the room who have equal number of acquaintances.

   (Hints: Each individual can have acquaintances in the range [1 .. 49]. Why?)

7. Problem from the text:
   - **Section 5.5:** 4, 5(a), 7(a), 8(a), (b), 10, 14, 20
   - **Supplementary problem:** 14, 24

PART A (contd.)

7. Consider n distinct numbers $a_1, a_2, \ldots, a_m$. Let m = min $\{a_1, a_2, \ldots, a_n\}$ and M = max $\{a_1, a_2, \ldots, a_n\}$. We define the gap of of two elements a_i and a_j to be $|a_i - a_j|$ if there does not exist any other element a_k with $a_i < a_k < a_j$, otherwise it is 0. Show that there exist two elements in $\{a_1, a_2, \ldots, a_n\}$ whose gap is at least (M-m)/(n+1).



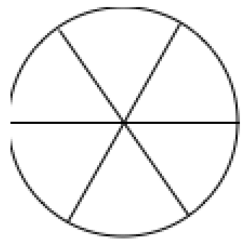
The gap between a_2 and a_6 is the largest; the gap between a_4 and a_6 is zero, since a_2 lies in between a_4 and a_6.

(Hint: Partition the interval [m .. M] into n+1 small sub-intervals, each of length (M-m)/(n+1).)

Discussed in the class.

PART B

1. Seven darts are thrown onto a circular dartboard of radius 10 units. Show that there will be two darts which are at most 10 units apart.



Ans: We first partition the board into sectors by drawing lines through the center with 60 degree angle apart. We will get 6 sectors as shown above. If seven points are thrown into 6 sectors, there will be a sector containing at least two points (by the pigeonhole principle). I have shown that any two points within a sector are at most 10 units apart.

2. 6 computers on a network are connected to at least 1 other computer. Show there are at least two computers that have the same number of connections.

Ans: Let x_i be the number of computers connected to computer i. Then the value of x_i is at least one, and at most 5. There are 6 computers (pigeons) and there are five different connections (holes) possible. By the pigeonhole principle, there are two computers with the same number of connections.

PART B (contd.)

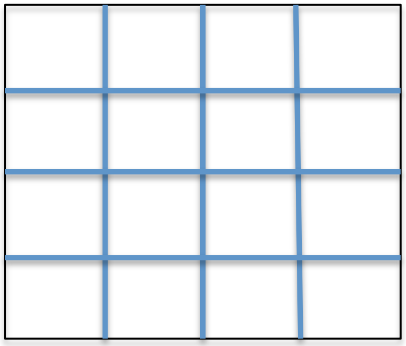
3. Consider 5 distinct points (x_i, y_i) with integer values, where i = 1, 2, 3, 4, 5. Show that the midpoint of at least one pair of these five points also has integer coordinates.
   - (Hints: We are looking for the midpoint of a segment from (a,b) to (c,d). The midpoint is ((a+c)/2, (b+d)/2). Note that the midpoint will be integers if a and c have the same parity: are either both even or both odd. The same is true for b and d.)

   Ans: When a pair of points (a,b) is picked up, then the parity of the pair is an element of {(odd,odd), (odd,even), (even,odd), (even,even)}. Thus if 5 pairs of points are selected, there will be two pairs of points with the same parity. This is by the principle of pigeonhole principle. Here 5 pairs of points are considered as pigeons and 4 possible parity outcomes as the pigeonholes.

PART B (contd.)

4. Suppose 49 points are placed, in a random way, into a square of side 1 unit. Prove that 4 of these points can be covered by a circle of diameter (1/√8). (Hints: The square should create 16 holes.)

Ans: The unit square is partitioned into 16 smaller squares of side length ¼. If 49 points are placed in 16 squares, by the generalized pigeon hole principle, there exists a square that contains at least 4 (ceiling(49/16)) points. The diameter of each square is (1/√8). Therefore, the above claim is true.



PART B (contd.)

5. Given m positive integers $a_1, a_2, \ldots, a_m$, show that there exists k and l with $0 \le k < l \le m$ such that $a_{k+1} + a_{k+2} + \ldots + a_l$ is divisible by m.

Ans: Let S_i denote the sum of first i terms, i.e $S_i = a_1 + a_2 + \ldots + a_i$, i=1,2, …, m. If m divides any one of S_i, we are done, since in this case, k=0 and l=i. If m does not divide any $S_{i,}$, there are m-1 non-zero remainders S_i mod m, i=1, 2, …,m. Thus we have m sums (pigeons) and m-1 possible remainders (pigeonholes). Therefore, there are two sums S_i and S_j such that S_i mod m= S_j mod m. Assuming i < j, we can show that $S_j - S_i$ is divisible by m. That is $a_{i+1} + a_{i+2} + \ldots + a_j$ is divisible by m. Here k=i, and l=j.