Homework 6
MACM 101
March 14, 2014
Date due: March 21, 2014.

PART A: Practice questions. Some of the questions in this part will be covered in the tutorials. You are strongly encouraged to work on these problems. You are not required to hand in the solutions to the problems in this part.

1. Problem from the text.
   (a) Section 4.2: 1, 11, 12, 16
   (b) Section 5.1: 4, 6, 13
   (c) Section 5.2: 1, 5, 8, 11,16
2. Suppose that a sequence $a_n, n = 0, 1, 2, ...$ is defined recursively by $a_0 = 1, a_1 = 7, a_n = 4a_{n-1} - 4a_{n-2}, n \geq 2$. Prove by induction that $a_n = (5n + 2)2^{n-1}\ \forall n \geq 0$.
3. Recursively define the set $S = \{n^2 | n \in \mathbb{Z}^+\}$, that is $S = \{1, 4, 9 \ldots\}$. What should we do to generate the elements in increasing sequence? This means that after generating 4^2, we need to generate 5^2.
   Hint: if x is in S, $x + 2x^{\frac{1}{2}} + 1$ is in S. Why?
4. Recursively define the set $T = \{2^n - 2 | n \in \mathbb{Z}^+\}$, $T = \{0, 2, 6, 14, 30, \ldots\}$.
5. Recursively define the set of strings $U = \{a^n bc^n | n \in \mathbb{N}\}$, that is $U = \{b, abc, aabcc, \ldots\}$.
6. Recursively define the set $V = \{x | x \in \mathbb{N} \wedge \lfloor \frac{x}{2} \rfloor$ is even $\}$, that is $V = \{0, 4, 8, 12, 16, \ldots\}$.
   (Hints: if x is in V, then $x + 4$ is in V.)
7. There are four different functions $f : \{0, 1\} \to \{0, 1\}$. List them all. Diagrams will suffice.
8. Consider the set $f = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x + 3y = 4\}$. Is this a function from $\mathbb{Z}$ to $\mathbb{Z}$? Explain.
9. A function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined as $f(m, n) = 3n - 4m$. Verify whether this function is injective and whether it is surjective.

Part B: Homework questions.

1. Give a recursive definition of the set of bit strings that are palindromes over the alphabet $\{0, 1\}$. Examples of palindromes: $\{\}, 0, 1, 00, 11, 000, 101, 010, 111, 0000, 10001$.

There are two types of palindrome, so we need two base cases: empty palindrome and x is a palindrome for each x in $\{0, 1\}$. The recursive step is: if a is a palindrome and x is a symbol, xax is a palindrome.

ans: Let S denote the set of palindromes.

Initialization: $S = \{\lambda, 0, 1\}$. Here λ indicates empty palindrome.
Recursion: If $x \in S$, then $0x0$ and $1x1$ also belong to S.

2. Give a recursive definition of the functions max and min so that $max(a_1, a_2, \ldots, a_n)$ and $min(a_1, a_2, \ldots, a_n)$ are the maximum and the minimum of n numbers $a_1, a_2, \ldots, a_n$, respectively.

   Ans:
   Let $M_i = \max\{a_1, a_2, \ldots, a_i\}$, and $m_i = \min\{a_1, a_2, \ldots, a_i\}$.
   Initialization:

   $$M_1 = a_1;\ M_2 = a_1 \text{ if } a_1 \geq a_2, \text{ otherwise } M_2 = a_2$$

   $$m_1 = a_1;\ m_2 = a_1 \text{ if } a_1 \leq a_2, \text{ otherwise } m_2 = a_2$$

   Recursion:

   $$M_n = \max\{M_{n-1}, a_n\} \text{ for } n \geq 3$$

   $$m_n = \min\{m_{n-1}, a_n\} \text{ for } n \geq 3$$

3. Suppose $A = \{0, 1, 2, 3, 4\}$, and $B = \{2, 3, 4, 5\}$ and $f = \{(0, 3), (1, 3), (2, 4), (3, 2), (4, 2)\}$. State the domain and range of f. Find $f(2)$ and $f(1)$.

   Ans: It is easy.

4. Suppose $f : \mathbb{Z} \to \mathbb{Z}$ is defined as $f = \{(x, 4x + 5) : x \in \mathbb{Z}\}$. State the domain, codomain and range of f.

   Ans:

   From the definition it is easy to say that $\mathbb{Z}$ is the domain and codomain of f.

   In order to answer for the range of f, consider an element y of codomain $\mathbb{Z}$. We are interested in determining x in the domain such that $f(x) = y$, i.e. $4x + 5 = y$. Thus $x = (y - 5)/4$. Since the domain is $\mathbb{Z}$, y should be such that $x = (y - 5)/4$ is an integer. Therefore, the range of f is $\{y \in \mathbb{Z} | (y - 5)/4 \text{ is an integer}\}$.

5. Consider the set $f = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : 3x + y = 4\}$. Is this a function from $\mathbb{Z}$ to $\mathbb{Z}$? Explain.

Ans: Proof is by contradiction. Suppose that f is not a function. As a result, there exist y_1 and y_2 in $\mathbb{Z}$, $y_1 \neq y_2$, and an $x \in \mathbb{Z}$ such that $f(x) = y_1$ and $f(x) = y_2$. This means $3x + y_1 = 3x + y_2$. This implies that $y_1 = y_2$, a contradiction. Therefore, f is a function.

6. Is the set $\theta = \{((x, y), (3y, 2x, x + y)) : x, y \in \mathbb{R}$ a function? If so, what is its domain, codomain and range?

   Ans: The domain of θ is $\mathbb{R} \times \mathbb{R}$. The codomain of θ is $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

   The range, say B, of θ is of course a subset of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$. However, we can write $B = \{(3y, 2x, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} | z = x + y\}$. This says that all the mapped points $(3y, 2x, x + y)$ lie on a plane $z = y + x$ in 3-dimensional space $\mathbb{R}^3$.

7. A function $f : \mathbb{Z} \to \mathbb{Z}$ is defined as $f(n) = 2n + 1$. Verify whether this function is injective and whether it is surjective.

   Ans:

   (a) $f(n) = 2n + 1$ **is injective:**
   We know that (using the contrapositive equivalent)

   $$f \text{ is one-to-one} \to (\forall x, y \in \mathbb{Z}, f(x) = f(y) \to x = y)$$

   Consider two arbitrary elements x and y in the domain of f. Now $f(x) = f(y)$ implies $2x + 1 = 2y + 1$. Therefore, $x = y$. Thus the above proposition is true, and therefore, f is injective.

   (b) $f(n) = 2n + 1$ **is not surjective:**

   Intuitively, this can be seen easily. Note that $2n + 1$ is an odd number for any $n \in \mathbb{Z}$. Therefore, even integers of the codomain can never be an image of f. This can be formally proved as follows. We know that f is surjective, then $\forall b \in \mathbb{Z}(codomain)\ \exists a \in \mathbb{Z}, f(a) = b$. Consider an arbitrary element b in the codomain $\mathbb{Z}$ of f. Let x be the pre-image of b. Therefore, by definition $2x + 1 = b$. This says that $x = (b - 1)/2$. Note that x is not an interger when b is an even integer. This implies that there is no pre-image of any even integer under f. Therefore, f is not an onto function.

8. Prove that the function $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$ defined by $f(x) = (\frac{x+1}{x-1})^3$ is bijective.

   Ans:

   (a) $f(x) = (\frac{x+1}{x-1})^3$ **is injective:**
   Consider the statement (using the contrapositive equivalent):

   $$f \text{ is one-to-one} \to (\forall x, y \in \mathbb{R} - \{1\}, f(x) = f(y) \to x = y)$$

Consider two arbitrary elements x and y in the domain of f. Now $f(x) = f(y)$ implies

$(\frac{x+1}{x-1})^3 = (\frac{y+1}{y-1})^3$.

i.e $\frac{x+1}{x-1} = \frac{y+1}{y-1}$

i.e. $1 + \frac{2}{x-1} = 1 + \frac{2}{y-1}$

i.e. $x = y$

Thus the statement is true, and therefore, the function is injective.

(b) $f(x) = (\frac{x+1}{x-1})^3$ **is srujective:**
We know that f is surjective implies $\forall b \in \mathbb{R} - \{1\}$ $(codomain)$ $\exists a \in \mathbb{R} - \{1\}$ $(domain), and f(a) = b$. Consider an arbitrary b in the codomain of f. Let a be a pre-image of b. Therefore,

$(\frac{a+1}{a-1})^3 = b$

i.e. $\frac{a+1}{a-1} = b^{\frac{1}{3}}$

i.e. $1 + \frac{2}{a-1} = b^{\frac{1}{3}}$

i.e. $a = 1 + \frac{2}{b^{1/3}-1}$.

a is not defined only when $b = 1$. But $b = 1$ is not an element of $\mathbb{R} - \{1\}$. This implies that $\forall b \in \mathbb{R} - \{1\}$, there is a pre-image under f. Therefore, f is an onto function.