Homework 5
MACM 101
March 3, 2014
Date due: March 12, 2014.

PART A: Practice questions. Some of the questions in this part will be covered in the tutorials. You are strongly encouraged to work on these problems. You are not required to hand in the solutions to the problems in this part.

1. Section 4.1: Problems $2, 14, 19, 23$
2. Show that $3^n \geq n + 2$ for all integers $n \geq 1$.
3. Suppose $A_1, A_2, \ldots, A_n$ are subsets of a universal set U, and integer $n \geq 2$. Prove that $\overline{A_1 \cup A_2 \cup \ldots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}$.
4. Use mathematical induction to prove the binomial theorem
   If n is a non-negative integer, then
   $$(x+y)^n = \binom{n}{0}x^0 + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n}y^n.$$
   You may find that you need the following ($n \geq k$):
   $$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$
5. The sequence $< 1, 1, 2, 3, 5, 8, 13, \ldots >$ is known as the Fibonacci sequence. Let F_i denote the the i^{th} element of the sequence. Thus $F_1 = 1, F_2 = 1, F_3 = 2$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Prove that
   (a) $\sum_{i=1}^{n} F_i = F_{n+2} - 1$.
   (b) $\sum_{i=1}^{n} F_{2i-1} = F_{2n}$
   (c) $\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1$
   (d) $F_n \leq (\frac{7}{4})^n$.
   (e) (Bonus) $F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$.
6. Prove that $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \geq 2(\sqrt{n+1} - 1)$.
7. If $n, k \in N$, and n is even and k is odd, $\binom{n}{k}$ is even.
8. Prove that $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$ whenever $n \geq 2$.
9. What is the largest number you cannot write as the sum of $6, 9$ or 20? That is what is the largest x such that $x \neq 6u + 9v + 20w$, where u, v and w are nonnegative integers.
10. Prove that $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$ whenever $n \geq 2$.

Part B: Homework questions.

1. Use Mathematical Induction to prove the validity of the following Rule of Inference for all integers $n \geq 1$:

$$\begin{array}{ccc} p_1 & \to & p_2 \\ p_2 & \to & p_3 \\ & & \\ \dots & \dots & \dots \\ & & \\ p_n & \to & p_{n+1} \\ & \neg p_{n+1} & \\ \hline & \neg p_1 & \end{array}$$

2. Suppose $A_1, A_2, \dots, A_n$ are subsets of a universal set U, and integer $n \geq 2$. Prove that $\overline{A_1 \cap A_2 \cap \dots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \dots \cup \overline{A_n}$.
3. Suppose n straight infinite lines lie on a plane in such a way that no two of the lines are parallel, and no three of the lines intersect at a single point. Show that this arrangement divides the plane into $\frac{n^2+n+1}{2}$ regions.
4. Prove that $3^0 + 3^1 + 3^2 + \dots + 3^n = \frac{3^{n+1}-1}{2}$.
5. (Bernoulli's inequality) Let $h > 1$. Use induction to show that

$$(1 + nh) \leq (1 + h)^n, n \geq 0.$$

6. Prove that $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.
7. Every integer $n \geq 14$ is expressible in the form $5a + 7b + 9c$ where a, b, c are nonnegative integers.