Solutions for Homework 5
MACM 101
Due: Mar 12, 2014

PART B:

1. Prove by mathematical induction

1) Basis Case: when n = 1, we have

$$\begin{array}{c} p_1 \rightarrow p_2 \\ \neg p_2 \\ \hline \therefore \neg p_1 \end{array}$$

Since $p_1 \rightarrow p_2 \Leftrightarrow \neg p_2 \rightarrow \neg p_1$, and together with $\neg p_2$, applying Modus Ponens we know $\neg p_1$ is valid.

2) Inductive steps:

Assume *S(k)* is valid, which is:

$$\begin{array}{c} p_1 \rightarrow p_2 \\ p_2 \rightarrow p_3 \\ \ldots \\ p_k \rightarrow p_{k+1} \\ \neg p_{k+1} \\ \hline \therefore \neg p_1 \end{array}$$

Then prove *S(k+1)* is valid:

$$\begin{array}{lc} & p_1 \rightarrow p_2 \\ & p_2 \rightarrow p_3 \\ & \ldots \\ & p_k \rightarrow p_{k+1} \\ (1) & p_{k+1} \rightarrow p_{k+2} \\ (2) & \neg p_{k+2} \\ \hline & \therefore \neg p_1 \end{array}$$

Consider statement (1) and (2). Since $p_{k+1} \rightarrow p_{k+2} \Leftrightarrow \neg p_{k+2} \rightarrow \neg p_{k+1}$, and $\neg p_{k+2}$, therefore $\neg p_{k+1}$ is true. Then we can substituted statement (1)(2) with $\neg p_{k+1}$, which makes *S(k+1)* = *S(k)*. As we assume *S(k)* is true, *S(k+1)* is true as well.

2. Prove by induction

1)Basis Case: n=2: There are only two sets, A_1 and A_2.

$$\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2} \quad \text{(De Morgan laws)}$$

2) Inductive Steps:

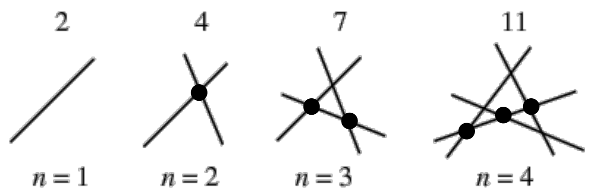
Assume S(k) : $\overline{A_1 \cap A_2 \cap \ldots \cap A_k} = \overline{A_1} \cup \overline{A_2} \cup \ldots \cup \overline{A_k}$ is true.

For S(k+1): $\overline{A_1 \cap A_2 \cap \ldots \cap A_k \cap A_{k+1}} = \overline{(A_1 \cap A_2 \cap \ldots \cap A_k) \cap A_{k+1}}$

$= \overline{(A_1 \cap A_2 \cap \ldots \cap A_k)} \cup \overline{A_{k+1}}$

$= \overline{A_1} \cup \overline{A_2} \cup \ldots \cup \overline{A_k} \cup \overline{A_{k+1}}$ (Plug in left side of S(k))

3. Hint: *R(n)* denotes the maximum number of regions divided by n lines, then $R(n+1) = R(n) + n + 1$. Why? Consider when you add a line, it divides all the regions through which it passes into two, thus adding one region for each region it passes through.



The number of regions that the new line passes through is the number of the intersections of the new line plus one. Since there are n lines in the plane, the $(n+1)_{th}$ new line will at most have n intersections with the n lines. Therefore there are $n+1$ new regions.

1) B.C. n=0, $R(0) = 1 = \frac{0^2+0}{2} + 1 = 1$, valid.

2) I.S. Assume $R(k) = \frac{k^2+k}{2} + 1$

Then $R(k+1) = R(k) + k + 1$

$$= \frac{k^2 + k}{2} + 1 + k + 1$$

$$= \frac{k^2 + 2k + 1 + k + 1}{2} + 1$$

$$= \frac{(k+1)^2 + k + 1}{2} + 1$$

4. Prove by induction

1) B.C. n=0, $3^0 = 1 \; and \; \frac{3^{0+1}-1}{2} = \frac{2}{2} = 1$, valid!

2) I.S. *S(k):* $3^0 + 3^1 + 3^2 + \cdots + 3^k = \frac{3^{k+1}-1}{2}$

S(k+1): $3^0 + 3^1 + 3^2 + \cdots + 3^k + 3^{k+1} = \frac{3^{k+1}-1}{2} + 3^{k+1}$

$$= \frac{3^{k+1}-1+2\times 3^{k+1}}{2} = \frac{3^{(k+1)+1}-1}{2}$$

5. (Bernoulli's inequality)

1) B.C. n=0, $1 \le 1$, valid!

2) I.S. Assume *S(k):* $(1+kh) \le (1+h)^k$ is valid,

For *S(k+1):* $(1+(k+1)h) = (1+kh) + h$

$$\le (1+h)^k + h$$

Now we need to prove $(1+h)^k + h \le (1+h)^{k+1}$

$$\Leftrightarrow 1 + \frac{h}{(1+h)^k} \le 1 + h$$

$$\Leftrightarrow \frac{h}{(1+h)^k} \le h$$

$$\Leftrightarrow 1 \le (1+h)^k \quad \text{(True! Since } h > 1)$$

6.

1) B.C. n=1, $\frac{1}{\sqrt{1}} = 1 \le 2\sqrt{1} = 2$

2) I.S. Assume *S(k)*: $\sum_{i=1}^{k} \frac{1}{\sqrt{i}} \le 2\sqrt{k}$

For *S(k+1)*: $\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} \le 2\sqrt{k} + \frac{1}{\sqrt{k+1}}$

Now prove $2\sqrt{k} + \frac{1}{\sqrt{k+1}} \le 2\sqrt{k+1}$

$$\Leftrightarrow 2\sqrt{k}\times\sqrt{k+1} + 1 \le 2\times(k+1)$$

$$\Leftrightarrow 2\sqrt{k(k+1)} \le 2k+1$$

$$\Leftrightarrow 4k(k+1) \le (2k+1)^2$$

$$\Leftrightarrow 4k^2 + 4k \le 4k^2 + 4k + 1 \quad \text{(True!)}$$

7. Prove by induction

1) B.C. S(14): $14 = 5\times0 + 7\times2 + 9\times0$;

S(15): $15 = 5\times3 + 7\times0 + 9\times0$;

S(16): $16 = 5\times0 + 7\times1 + 9\times1$;

S(17): $17 = 5\times2 + 7\times1 + 9\times0$;

S(18): $18 = 5\times0 + 7\times0 + 9\times2$;

2) I.S. Suppose *S(k)*: k = $5a + 7b + 9c$;

For *S(k+5)* : $k + 5 = 5a + 7b + 9c + 5 = 5(a + 1) + 7b + 9c$;

Consequently, if k is expressible, then *k+5* is also expressible, and since we discussed five consecutive basis cases, we will cover all the integers that larger than 14.