MACM 101 (Homework 4)
February 24, 2014
Date Due: March 3, 2014.

Part A: Practice questions. You are strongly encouraged to work on these problems. You are not required to hand in the solutions to these problems.

1. Problems from the text.
   (a) Section 3.1: $8, 10, 17, 18$
   (b) Section 3.2: $6(bc), 8, 14(c)$
   (c) Section 3.3: $4, 6$
   (d) Section 3.4: $6, 8, 10, 16$
2. Use the method of contrapositive proof to prove the following statements. (Try first the direct proof. You will notice that contrapositive proof is easier.)
   (a) $\forall a, b \in Z$: If $a^2(b^2 - 2b)$ is odd, then a and b are odd.
   (b) $\forall a, b, c \in Z$: If a does not divide bc, then a does not divide b.
3. Use the fact that any odd number is of the form $8k \pm 1$ or $8k \pm 3$ (k is an integer) in order to give a direct proof of the claim that the square of any odd number leaves remainder 1 upon division by 8. Use this to show that 2001 is not the sum of three odd numbers.
4. Let s be a positive integer. Prove that the closed interval $[s, 2s]$ contains a power of 2.
   {Hints: Use proof by cases. Consider two cases: (a) s is a perfect power of 2, and (b) s is not a perfect power of 2, i.e. $2^k < s < 2^{k+1}$ for some integer k.}
5. Suppose A, B and C are subsets of the universe U. Prove that
   (a) if $A \subseteq B$ then $A \cap C \subseteq B \cap C$.
   (b) if $A \subset B$ then $\overline{B} \subseteq \overline{A}$.
   (c) if $A \subset C$ and $B \subseteq C$ then $A \cup B \subseteq C$.
6. Determine if the following statements are true or false:
   (a) A and $B - A$ are disjoint.
   (b) $(A \subseteq \overline{B} \wedge \overline{A} \subseteq \overline{B}) \Rightarrow B = \Phi$.

Part B: Homework questions.

1. Prove, by arguing by contradiction, that $\sqrt{3}$ is an irrational number.
   Ans: Let $p : \sqrt{3}$ is irrational. We assume $\neg p$ (i.e. $\sqrt{3}$ is rational). Therefore, there exist two integers a and b, not both even, otherwise we simplify, such that $\sqrt{3} = \frac{a}{b}$. Squaring we get $a^2 = 3b^2$. We notice that both a and b have to be odd. Why? If b is even, b^2 is even. If b^2 is even, $3b^2$ is even. If $3b^2$ is even, a is even. This is a contradiction, since both a and b can't be even.

   Thus we have both a and b odd. Therefore, we can write $a = 2k+1$ and $b = 2k'+1$ for some integer k and k'. Thus

$$\begin{aligned} 0 &= a^2 - 3b^2 \\ 0 &= (2k+1)^2 - 3(2k'+1)^2 \\ 0 &= 4k^2 + 4k - 12k'^2 - 12k' - 2 \\ 0 &= 2(k^2 + k - 6k'^2 - 6k') - 1 \end{aligned}$$

   The right hand side is an odd non-zero number - it is a contradiction. Thus, $\sqrt{3}$ is a rational number leads to a contradiction. Therefore, $\sqrt{3}$ is an irrational number.

2. Prove, by arguing by contradiction, that there are no integers, a, b, c, d such that

$$x^4 + 2x^2 + 2x + 2 = (x^2 + ax + b)(x^2 + cx + d)$$

   Ans:

$$\begin{aligned} x^4 + 2x^2 + 2x + 2 &= (x^2 + ax + b)(x^2 + cx + d) \\ &= x^4 + (a+c)x^3 + (d+b+ac)x^2 + (ad+bc)x + bd \end{aligned}$$

   Thus

$$bd = 2, ad + bc = 2, d + b + bc = 2, a + c = 0$$

   Assume a, b, c, d are integers which satisfy the above identities. Since $bd = 2$, b and d must be of opposite parity (one odd, the other even). But then $d + b$ must be odd, and since $d + b + bc = 2$, bc must be odd, meaning that both b and c are odd. Therefore, d is even. This means that ad is even. Therefore, $ad+bc$ is odd. But we notice that $ad+bc = 2$ which says that $ad+bc$ is even. Thus we arrive at a contradiction: $ad+bc$ is both odd and even. Hence a, b, c, d can't be all integers.

3. Use the method of contrapositive proof to prove the following statements. (Try first the direct proof. You will notice that contrapositive proof is easier.)

(a) $\forall n \in Z$. If n^2 is even, then n is even.
Ans: Consider the open statements $p(n) : n^2$ is even, and $q(n) : n$ is even. The contrapositive of the above statement is $\forall n\ \neg q(n)\ \rightarrow\ \neg p(n)$. If $\neg q(n)$ is true, i.e. n is odd, clearly n^2 is odd. Therefore $\neg p(n)$ is true.

(b) $\forall x \in R$: If $x^3 - x > 0$ then $x > -1$.
Ans: Consider the open statements $p(x) : x^3 - x > 0$ and $q(x) : x > -1$. The contrapositive of the above statement is $\forall x\ \neg q(x)\ \rightarrow\ \neg p(x)$ Clearly, $x \leq -1(\neg\ q(x)$ is true) implies that $x^3 - x \leq 0$ ($\neg\ p(x)$ is true). The contraposition formula is very easy to prove here. You should select the proof method that is appropriate for the problem at hand.

4. Describe each of the following sets in the format $\{x|\text{property of x}\}$.

(a) $A = \{\ 0, 2, 4, 6, 8, ...\}$
Ans: $A = \{\ 2x\ |\ x\ \in N\ \}$

(b) $B = \{1, 5, 9, 13, ...\}$
Ans: $B = \{\ 4x + 1\ |x\ \in N\ \}$

(c) $C = \{1, 1/2, 1/3, 1/4, ...\}$
Ans: $C = \{\ \frac{1}{x}\ |\ x\ \in N^+\ \}$

5. Let U be the universal set and let $A, B, C \subseteq U$. Using the properties of union, intersection and complement and known set laws, simplify the followin:

(a) $\overline{A \cap U} \cup \overline{A}$
Ans: $\overline{A \cap U} \cup \overline{A} = \overline{A} \cup \overline{A} = \overline{A}$.

(b) $(A \cap B) \cap \overline{A}$.
Ans: $(A \cap B) \cap \overline{A} = \overline{\overline{A \cap B} \cup A} = \overline{\overline{A} \cup \overline{B} \cup A} = \overline{U \cup B} = \overline{U} = \Phi$.