Homework 3-1
MACM 101
February 8, 2014
Date due: February 22, 2014.
Feb. 26: Solution to the homework part

Part A: The following problems are for your practice only. You are strongly encouraged to work on these problems. You don't need to hand in the solutions to these problems. None of these problems will be considered for the marking.

1. Section 2.4: $2b, 4abcd, 8abcd, 12, 18, 26$
2. Section 2.5: $6, 10$
3. Find a counterexample, if possible, to the following quantified statement, where the universe for all variables consists of all integers
$$\exists x \; \exists y \; (x^2 + y^2 = 15)$$
4. Brown, Johns and Landau are charged with bank robbery. The thieves escaped in a car that was waiting for them. At the inquest Brown stated that the criminals had escaped in a blue Buick; Johns stated that it had been a black Chevrolet, and Landau said that it had been a Ford Granada and by no means blue. It turned out that wishing to confuse the Court, each one of them only indicated correctly either the make of the car or only its color. What color was the car and of what make?
5. Explain whether the following assertion is true and negate it. The universe for both m and n is $\mathbb{Z}$.
$$\forall n \; \exists m \; (n^2 > 4n \to 2^n > 2^m + 10)$$

Part B: Homework questions.

1. Rewrite the following statement so that negations appear only within predicates (that is, no negation is outside a quantifier or an expression involving logical connectives)
$$\neg \; (\forall x((\forall y \; \exists z \; p(x) \wedge q(y,z)) \oplus (\forall y \; \forall z \; (p(z) \vee q(z,y)))))$$
Ans:
We know that $a \oplus b$ is equivalent to $(\neg a \wedge b) \vee (a \wedge \neg b)$.
Let $t(x) = ((\forall y \; \exists z \; p(x) \wedge q(y,z))$, and $r = (\forall y \; \forall z \; (p(z) \vee q(z,y)))$. The statement in the question is equivalent to $\neg(\forall x(t(x) \oplus r))$, which

is equivalent to $\neg(\forall x((\neg t(x) \wedge r) \vee (t(x) \wedge \neg r)))$. We take the negation inside and get $\exists x((t(x) \vee \neg r) \wedge (\neg t(x) \vee r)))$.

Now $\neg t(x)$ is equivalent to $\exists y \forall z \neg p(x) \vee \neg q(y,z)$, and $\neg r$ is equivalent to $\exists y \exists z(\neg p(z) \wedge \neg q(z,y))$.

2. For each quantified formula that follows: find a universe U and predicates A, B, P in which the formula is true and U, A, B, P in which it is false.
   (a) $\forall x\ ((A(x) \vee B(x)) \wedge \neg(A(x) \wedge B(x)))$
   Ans: Let Z be the universe. Then
   TRUE: Open statements $A(x) : x$ is odd; $B(x) : x$ is even.
   FALSE: $A(x) : x$ is odd; $B(x) : x^2$ is odd.
   (b) $\forall x\ \forall y\ (P(x,y) \rightarrow P(y,x))$
   Ans: Universe is Z^+.
   TRUE: $P(x,y) : x * y > 0$.
   FALSE $P(x,y) : x/y > 1$.
   (c) $\forall x\ A(x) \rightarrow (\forall x\ B(x) \rightarrow (\forall x\ (A(x) \rightarrow B(x))))$
   Ans:
   TRUE: $A(x) : x$ is even. Universe is Z. Thus $\forall x A(x)$ is false. This makes the statement true.
   FALSE: In order to make the statement false, we have to make $\forall x A(x)$ and $\forall x B(x)$ to be true, and at the same time make $\forall x(A(x) \rightarrow B(x))$ false. This is not possible if the universe is the same for both $A(x)$ and $B(x)$.

3. For the following formulas, let the universe be $\mathbb{R}$. Translate each of the following sentences into a formula using quantifiers.
   (a) There is no largest number.
   Ans: $\neg(\exists m \forall n(m > n)$. Universe : R;

   (b) There is no smallest positive number.
   Ans: $\neg(\exists x \forall n(x < n) \wedge (x > 0))$.
   (c) Between any two distinct numbers, there is a third number not equal to either of them.
   Ans: $\forall x, z \exists y((x \neq z) \wedge ((x < y < z) \vee (z < y < x))$.

4. A, B, C, D ran a race. Asked how they made out, they replied:
   - A: " C won; D was second."
   - B: " C was second; D was third."
   - C: " D was last; A was second."

   If each of the individuals made one and only one true statement, who won the race?

{ hints: Given the propositions "p: C won"; and "q: D was second". From the reply of A we can say that $p \vee q$ is true. }

Ans: Consider the following propositions.

- $a_2 : A$ was second
- $c_1 : C$ won
- $c_2 : C$ was second
- $d_2 : D$ was second
- $d_3 : D$ was third
- $d_4 : D$ was last.

Since each individual gave one true statement, we have

$$(c_1 \vee d_2) \wedge (c_2 \vee d_3) \wedge (d_4 \vee a_2) \text{ is true.}$$

Using the distribution rule we can write:

$$(c_1 \wedge c_2 \wedge d_4) \vee (c_1 \wedge c_2 \wedge \wedge a_2) \vee (c_1 \wedge d_3 \wedge d_4) \vee (c_1 \wedge d_3 \wedge a_2) \vee (d_2 \wedge c_2 \wedge d_4) \vee (d_2 \wedge c_2 \wedge a_2) \vee (d_2 \wedge d_3 \wedge d_4) \vee (d_2 \wedge d_3 \wedge a_2)$$

is a true value. Now $c_1 \wedge c_2, d_3 \wedge d_4, c_2 \wedge d_2, d_2 \wedge a_2$ are all **false**. The only surviving term is $(c_1 \wedge d_3 \wedge a_2)$. Therefore, C won, D was third and A was second.

5. Explain whether the following assertion is true and negate it. The universe for both m and n is $\mathbb{Z}$.

$$\forall n \; \exists m \; (n > 3 \rightarrow (n+7)^2 > 49 + m)$$

Ans: Simplifying $(n+7)^2 > 49 + m$, we get $n^2 + 14n > m$. The given assertion can now be written as $\forall n \; \exists m \; (n > 3 \rightarrow n^2 + 14n > m)$. Since $n > 3$, $n^2 + 14n$ is at least $4^2 + 14 * 4$ which is 72. Thus the assertion is true for any m less than 72.

Negation of the assertion: $\neg(\forall n \; \exists m \; (n > 3 \rightarrow (n+7)^2 > 49 + m))$.

We first replace the implication by $\forall n \; \exists m(\neg(n > 3) \vee ((n+7)^2 > 49+m))$.

Now it is easy to take the negation inside: $\exists n \forall m((n > 3) \wedge (n+7)^2 \leq 49 + m)$.