# Homework #1 Solutions

January 28, 2014

## Section 1.1 and 1.2

### Question 2

There are five vowels, (A, E, I, O, U), and five even digit, (0, 2, 4, 6, 8), so by the rule of product there are $5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^6$ different plates where the first two letters are vowels and the last four digits are even.

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### Question 4

**a.**

The positions president, vice president, secretary, and treasurer are all distinct, so by the rule of product there are $10 \times 9 \times 8 = P(10, 4) = 5040$ different slates.

**b.**

**(i)** The president has to be chosen from the three physicians. For the positions of vice-president, secretary, and treasurer we will have 9, 8, and 7 options respectively. Therefore, from the rule of product the total number of slates the board can present is $3 \times 9 \times 8 \times 7 = 1512$.
**(ii)** Assume one of the physicians is to occupy the position of president. As there can only be one physician on the slate, and by the rule of product, the other positions can be filled in $7 \times 6 \times 5$ different ways. So, the total number of slates having one physician as president is $3 \times 7 \times 6 \times 5 = 630$. The same is true if the physician was to be assigned to the position of vice-president, secretary, or treasurer. So, the total number of slates that include exactly one physician is $4 \times (3 \times 7 \times 6 \times 5) = 2520$.
**(iii)** The number of slates not including any physicians is $7 \times 6 \times 5 \times 4 = 840$. From part (i), the total number of slates possible is 5040. Consequently, $5040 - 840 = 4200$ slates include at least one physician nominated for one of the four offices.

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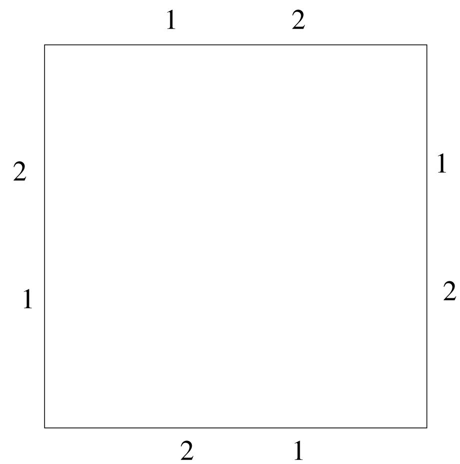
### Question 16

**a.**

When repetitions are allowed, there can be at most $40^{25}$ distinct messages.

**Figure 1:** Seats available to A




**b.**

Knowing that 10 of the letters can only appear as the first and/or last symbol (meaning that any of these letters <u>can</u> appear as both first and last symbol), we have 40 choices for each of these two symbols. For the remaining 23 symbols we will have only 30 choices, so by the rule of product, we can have $40^2 \times 30^{23}$ distinct messages under these new rules.

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## Question 34

It would be easier to consider different cases:

| | | |
|---|---|---|
| | $(4!/2!)$ | no 7's and two 3's |
| $+$ | $(4!)$ | one 7 and one 3 |
| $+$ | $2 \times (4!/2!)$ | one 7 and two 3's (Note that for the fourth digit we have the choice of either 1 or 8) |
| $+$ | $2 \times (4!/2!)$ | two 7's and and one 3 (Similar to the previous case) |
| $+$ | $(4!/2!)$ | two 7's and no 3's |
| $+$ | $(4!/(2! \times 2!))$ | two 7's and two 3's |
| $=$ | 102 | |

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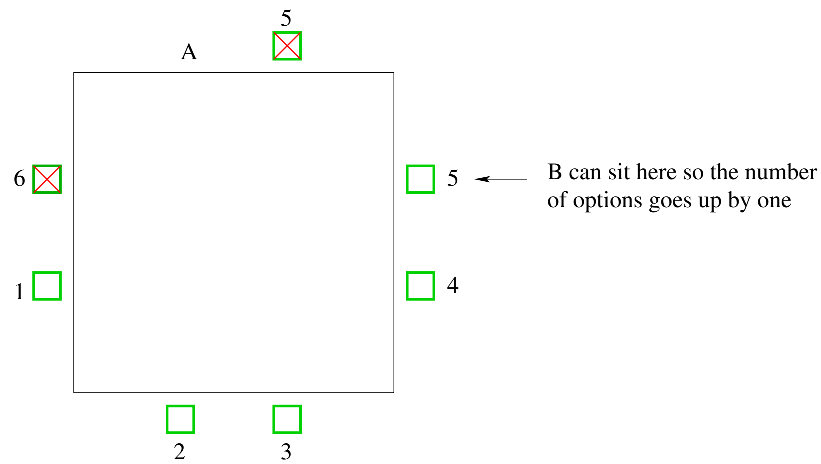
## Question 36

**a.**

First, we locate $A$. As shown in Figure 1, $A$ can either sit on a chair labeled 1, or labeled 2. Now if $A$ is sitting on a chair labels 1 (it does not make a difference which chair labels 1, we can just assume $A$ is sitting on the top one), then there will be a person to the left of $A$ on the same side of the table. In this case, there are 7! possible seating arrangements. If $A$ is sitting on a chair labeled 2, then a person

**Figure 2:** Seats available to B




can sit to the right of $A$ on the same side of the table. Likewise, there are 7! ways people can be seated at the table in this case as well. So, there are $2 \times 7! = 10080$ seating arrangements in total.

**b.**

Now assume $A$ is sitting on the top chair labeled 1. $B$ cannot sit on any of the two chairs crossed of in Figure 2. Therefore, if we start locating people from the crossed chair to the right of $A$ (on the other side of the table), the number of possibilities at each location is as shown in Figure 2: $6 \times 5 \times 5! = 3600$. The same holds if $A$ is seated at a chair labeled 2, so the total number of seating arrangements in which $A$ and $B$ do not sit together is $2 \times 3600 = 7200$.

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## Section 1.3

### Question 4

**a.**

Each of the 6 dots can be raised or not, so for each, we have 2 possibilities. But, any symbol in Braille system myst have at least one raised dot, so the only choice that is not allowed is when none of the dots is raised. Therefore, there are $2^6 - 1 = 63$ distinct symbols.

**b.**

We should choose three dots out of six, so the number of symbols with exactly three raised dots is $\binom{6}{3}$.

**c.**

Either two, four, or all six of the dots should be raised, so the number of symbols is equal to $\binom{6}{2} + \binom{6}{4} + \binom{6}{6} = 31$.

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## Question 8

**a.**

**Flush:** There are four suits to choose from. Once the suit is chosen, we should pick 5 cards of that suit, so the number of possibilities is $\binom{4}{1}\binom{13}{5}$.

**b.**

**Four aces:** All fours aces must be chosen. From the remaining 48 cards, we still need to pick one, so the answer is $\binom{4}{4}\binom{48}{1}$.

**c.**

**Four of a kind:** We should get four of the same rank. There are 13 ranks to pick from. Then, all the card of that rank should be chosen, and like the previous case, we should pick one last card from the remaining 48. $\binom{13}{1}\binom{4}{4}\binom{48}{1}$.

**d.**

**Three aces and two jacks:** $\binom{4}{3}\binom{4}{2}$.

**e.**

**Three aces and a pair:** Like the previous case, we pick three of four aces. For the pair (two cards of the same rank), we should first pick a rank. We cannot chose aces now, so there are twelve remaining ranks to choose from. Then, we just chose two of that rank. The number of ways we can get three aces and a pair is $\binom{4}{3}\binom{12}{1}\binom{4}{2}$.

**f.**

**A full house:** That is, three cards from one rank ($\binom{13}{1}\binom{4}{3}$), and two from another ($\binom{12}{1}\binom{4}{2}$). By the rule of product we have $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3,744$ possibilities.

**g.**

**Three of a kind:** We want to get three of the same kind, but avoid getting a full house, so the other two cards should not be of the same rank. There are a couple of ways to answer this question:
**(i)** First choose a rank and pick three cards from it. Then for card number 4 choose one from the remaining 48, throw out the three other cards of the same rank as card number 4, and then for card number 5 choose a card from the remaining 44. This way, we are counting each draw of cards twice as the fourth and fifth card might come in the reverse order as well. While they are the same, we are counting them as two different possibilities. So, we cancel our over counting by dividing by two: $\binom{13}{1}\binom{4}{3}\binom{48}{1}\binom{44}{1}/2 = 54,912$.

**(ii)** First choose a rank and pick three cards from it. Then choose two other cards from the remaining 48. This way, we have counted the ways we can get either three of a kind or a flush. So if we subtract the number of ways we can get a flush, we should have the number of possible draws for three of a kind: $\binom{13}{1}\binom{4}{3}\binom{48}{2} - 3,744 = 54,912$.

**(iii)** First choose a rank and pick three cards from it. Then, choose two our of twelve remaining ranks, and pick one card from each: $\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1} = 54,912$.

**h.**

**Two pairs:** Choose two ranks first. Then, pick two cards from each and throw the rest of the cards from those ranks away. Finally we need one more can, which we can choose from the remaining 44. So, the total number of two pairs is $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}$.

**Note:** There are other possible ways of counting the number of two pair draws as well.

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## Question 12

**a.**

Assume we first give three books to the first child, then another three to the second child, and so forth. The first set of three books can be selected in $\binom{12}{3}$ ways. The second set of three books can be chosen in $\binom{9}{3}$ ways. The third set can be chosen in $\binom{6}{3}$ ways, and finally the fourth set can be selected in $\binom{3}{3}$ ways. Therefore, the books can be distributed among the children in $\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3} = 12!/[(3!)^4]$ different ways.

**b.**

In a similar way, we can distribute the books in $\binom{12}{4}\binom{8}{4}\binom{4}{2}\binom{2}{2} = 12!/[(4!)^2(2!)^2]$ possible ways.

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## Question 18

**a.**

$10!/(4!3!3!)$. Equivalently, we could say we first choose the positions for four 0's, then positions for three 1's, and finally the positions for three 2's: $\binom{10}{4}\binom{6}{3}\binom{3}{3} = 10!/(4!3!3!) = 4200$.

**b.**

At least eight 1's means that we get either exactly eight 1's, or exactly 9 1's, or exactly ten 1's:

$$
\begin{array}{lll}
 & \binom{10}{8} 2^2 & \text{for each of the remaining two positions we have two options (either 0 or 2)} \\
+ & \binom{10}{9} 2^1 & \text{for the last digit we have two options (either 0 or 2)} \\
+ & \binom{10}{10} & \\
= & 201 &
\end{array}
$$

**c.**

We consider different cases we can get a weight 4 (i.e., the sum of digits equal to 4):

$$
\begin{array}{lll}
 & \binom{10}{4} & \text{four 1's, six 0's} \\
+ & \binom{10}{2}\binom{8}{1} & \text{two 1's, one 2, seven 0's} \\
+ & \binom{10}{2} & \text{two 2's, eight 0's} \\
= & 615 &
\end{array}
$$

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## Problem

We start form $B(3)$ as shown in Figure 3. The polygon is a triangle, so we are done and there is only one possible triangulation.

Next step is $B(4)$ (See Figure 4). Here, we fix one edge as the *base*. Assume we have selected the bottom edge as the base. Now we should choose a vertex to form a triangle upon this base. For a quadrilateral as the one in Figure 4, we have two choices, namely the shaded triangles depicted in Case 1 and Case 2. In Case 1, we have a triangle to the right of the shaded area, and fortunately, we know how to calculate the number of triangulations form $B(3)$ (although it is quite straightforward that the number of triangulations for a single triangle is one, this trick of referring to the previously known results comes in handy shortly). Consequently, we can count the number of possible triangulations for each of the cases.

$$
\begin{array}{lll}
 & B(\varnothing) \times B(3) & \text{Case 1: nothing to the left of the shaded area and a triangle to the right of it} \\
+ & B(3) \times B(\varnothing) & \text{Case 2: a triangle to the left of the shaded area and nothing to the right of it} \\
= & 2 &
\end{array}
$$

Here, $\varnothing$ indicates "nothing", and we let $B(\varnothing) = 1$. One can think of it as follows: the number of ways to triangulate "nothing" is one – not to triangulate it at all!

**Figure 3:** Calculating $B(3)$

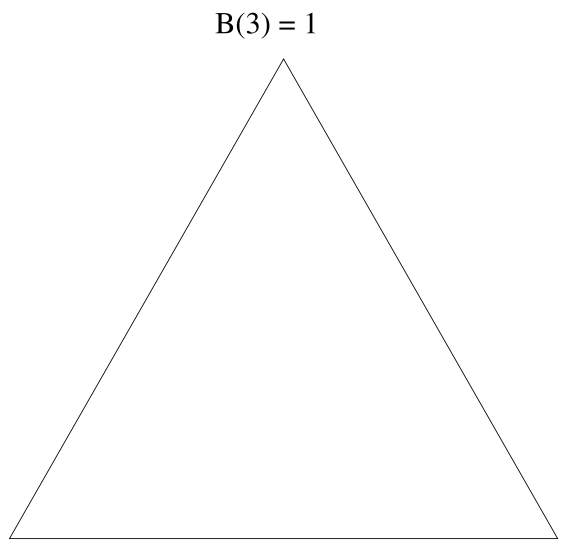



**Figure 4:** Calculating $B(4)$

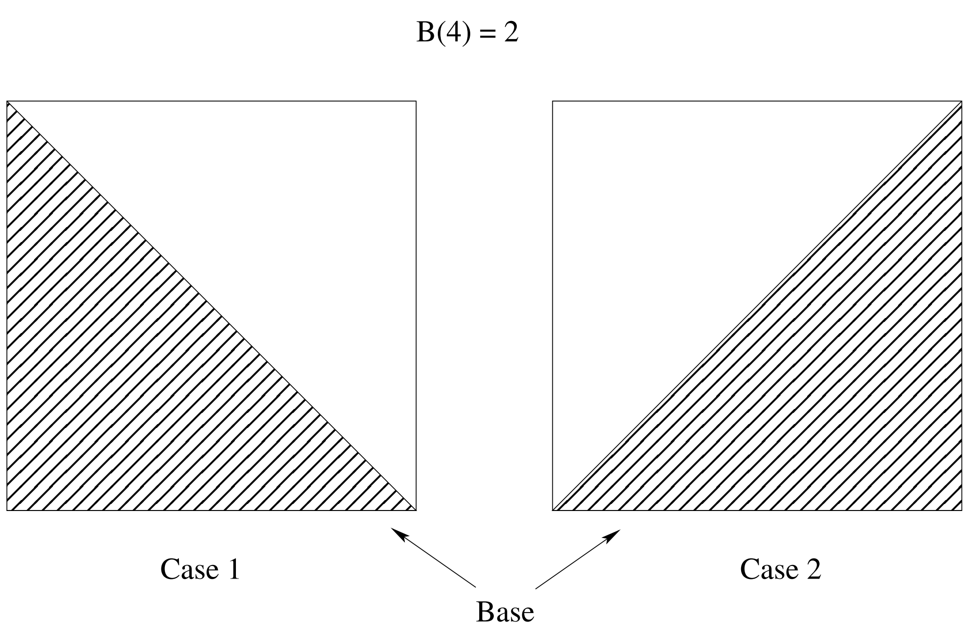
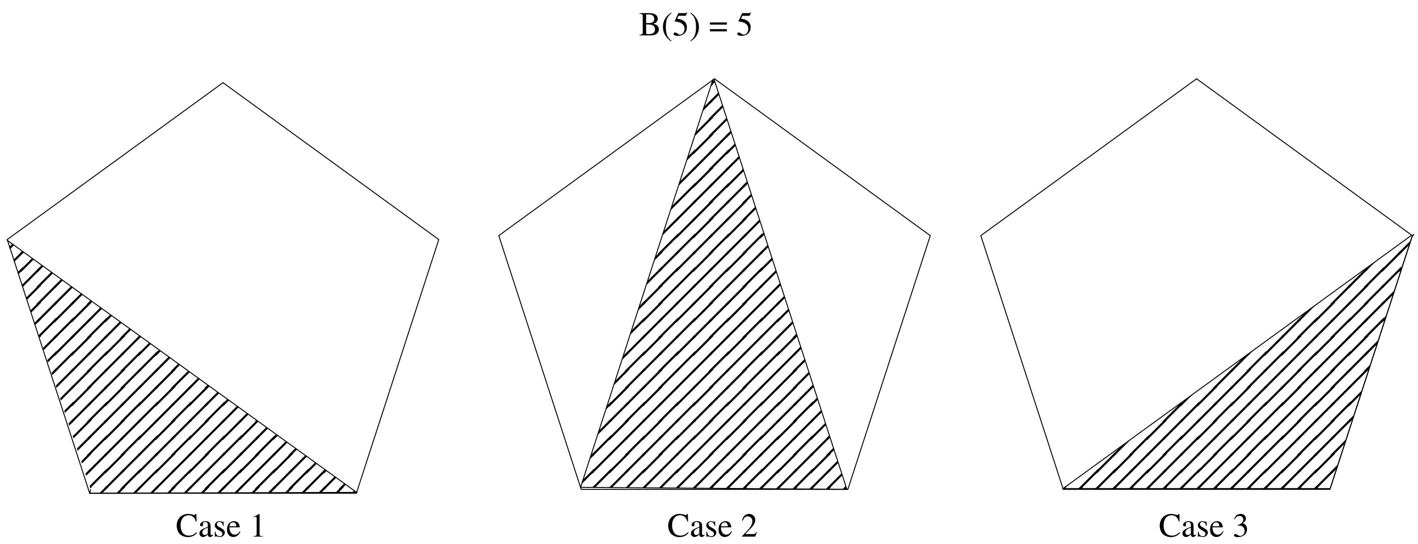


**Figure 5:** Calculating $B(5)$




Next, we calculate $B(5)$. We use the same idea as described above. Henceforth, we assume the bottom edge is the base. We get three cases here as depicted in Figure 5. For all these cases, we can use the previous results we have had so far:

| | | |
|---|---|---|
| | $B(\varnothing) \times B(4)$ | Case 1: nothing to the left of the shaded area and a quadrilateral to the right of it |
| $+$ | $B(3) \times B(3)$ | Case 2: a triangle to the left of the shaded area and another triangle to the right of it |
| $+$ | $B(4) \times B(\varnothing)$ | Case 3: a quadrilateral to the left of the shaded area and nothing to the right of it |
| $=$ | 5 | |

Finally, we get to $B(6)$. We can distinguish between four cases (Figure 6).

| | | |
|---|---|---|
| | $B(\varnothing) \times B(5)$ | Case 1: nothing to the left of the shaded area and a pentagon to the right of it |
| $+$ | $B(3) \times B(4)$ | Case 2: a triangle to the left of the shaded area and a quadrilateral to the right of it |
| $+$ | $B(4) \times B(3)$ | Case 3: a quadrilateral to the left of the shaded area and a triangle to the right of it |
| $+$ | $B(5) \times B(\varnothing)$ | Case 4: a pentagon to the left of the shaded area and nothing to the right of it |
| $=$ | 14 | |

**Figure 6:** Calculating $B(6)$