Class notes on sections 4.2, 5.1, 5.2 and 5.3

- Topics covered:
  - Recursive definition
  - Rudimentary discussion on relations
  - Functions: injective, surjective and bijective

Recursive Definitions

- Recursively defined sequence
  - Consider the Fibonacci sequence:

    $\{F_n\} = 1, 1, 2, 3, 5, ...$

    Here $F_1=1, F_2=1, F_3=2, ...$

    Recursive Definition of $\{F_n\}$

    - Initialization: $F_1 = 1, F_2=2$
    - Recursion $F_n = F_{n-1} + F_{n-2}, n >= 3$

Recursive Definitions (contd.)

- Recursively defined functions
  - Consider the factorial function:

    n! = 1.2.3. ….(n-2)(n-1)n

    Here 0!=0, 1!=1, 2!=2

    Recursive Definition of n!

    - Initialization: 1! = 1
    - Recursion n! = n.(n-1)!, n >= 2

Recursive Definitions (contd.)

- **Recursively defined functions**
  - Consider the binomial function:
    $C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$
    Recursive definition of $C(n,k)$

    $C(n,k) = 0$ if $k < 0$ or $k > n$.

    $C(n,k) = 1$ if $k = 0; n = 0$

    $C(n,k) = C(n-1,k) + C(n-1,k-1)$, otherwise.

Recursive Definitions (contd.)

- Recursively defined mathematical notations
  - Consider the sum

    $S_n = a_1+a_2+ .... +a_n$

    Here $S_0=0$, $S_1 = a_1$, $S_2=a_2$

    Recursive Definition of S_n

    - Initialization: $S_1 = a_1$
    - Recursion $S_n= S_{n-1} + a_n$, $n >= 2$

  - Similar definitions can be described for the product

    $P_n= a_1a_2.....a_n$ where $P_1 = a_{1.}$

Recursive Definitions (contd.)

- Recursively defined sets (defining the elements of a set recursively)
  - Consider the set S of prices (cents) payable using quarters and dimes.

    Recursive Definition of S

    - Initialization: $0 \varepsilon S$
    - Recursion If $x \varepsilon S$, $x+10 \varepsilon S$ and $x+25 \varepsilon S$.

    Note that only the distinct elements of S are kept.
  - Recursive defn. of +ve and –ve powers of 2
    - Initialization: $1 \varepsilon T$
    - IF $x \varepsilon T$, $2x \varepsilon T$ and $x/2 \varepsilon T$.

Recursive Definitions (contd.)

- Recursively defined sets (defining the elements of a set recursively)
  - Consider the power set of A.

$\mathcal{P}(\mathrm{A}) = \{\{\}\}$ when $A = \{\}$

$\mathcal{P}(\mathrm{A}) = \mathcal{P}(\mathrm{A}\text{-}\{\mathrm{x}\}) \cup \mathcal{P}(\mathrm{X} \cup \{\mathrm{x}\} | \mathrm{X} \in \mathcal{P}(\mathrm{A} - \{\mathrm{x}\})$ when A is not empty.

Recursive Definitions (contd.)

- Recursively defined character strings
  - Defn: A string is a finite sequence of 0 (null string) or more letters of alphabet Σ.

    For binary strings the alphabet set Σ = {0,1}.

    Defining binary strings B recursively:

    - Initialization: {} ε B
    - IF u ε B, u || ’0’ ε B and u || ’1’ ε B.

    Here || indicates concatenation.

Recursive Definitions

- Factorial

$$n! = \begin{cases} 1, \text{ if } n = 0 \\ n(n-1)!, \text{ if } n \geq 1 \end{cases}$$

- Fibonacci sequence

$$f(n) = \begin{cases} n, \text{ if } k = 0 \text{ or } 1 \\ f(n-2) + f(n-1), \text{ if } k \geq 2 \end{cases}$$

- Binomial Coefficients "*n*-choose-*k*"

$$C(n,k) = \begin{cases} 0, \text{ if } k < 0 \text{ or } k > n \\ 1, \text{ if } k = n = 0 \\ C(n-1,k-1) + C(n-1,k), \text{ otherwise} \end{cases}$$

- Addition of non-negative integers

$$m + n = \begin{cases} m, \text{ if } n = 0 \\ m+1, \text{ if } n = 1 \\ (m + (n-1)) + 1, \text{ if } n > 1 \end{cases}$$

- Summation Notation

$$\sum_{i=1}^{n} a_i = \begin{cases} 0, \text{ if } n = 0 \\ \sum_{i=1}^{n-1} a_i + a_n, \text{ if } n > 0 \end{cases}$$

- Product Notation

$$\prod_{i=1}^{n} a_i = \begin{cases} 1, \text{ if } n = 0 \\ \left(\prod_{i=1}^{n-1} a_i\right) \cdot a_n, \text{ if } n > 0 \end{cases}$$

Applications of Recursions

- Example: Find the recurrence for the number of n digit binary sequences with no pair of consecutive 1's.
  - Let A(n) denote the number of n digit binary sequences with no pair of consecutive 1s.
  - To write A(n) we condition on the last digit. If it is 0, the number of such sequence is A(n-1). If it is 1, the penultimate digit must be 0, and the number of such sequences sought is A(n-2).
  - Thus A(n) = A(n-1)+A(n-2) (inductive step)
  - Basis step: A(1) = 2; A(2) = 3

Relations
(sections 5.1)

- Let A and B are sets; A x B = {(a,b) | a ε A and b ε B};
  (a,b) are ordered pairs, also known as 2-tuple.
- The universes of A and B could be different.
- R x R = {(x,y)| x,y ε R} is the two-dimensional real plane.
- Binary relation:
  - For sets A and B, any subset of AxB is a binary relation from A to B. Any subset of A x A is called binary relation on A.
  - A = Z^+; {(x,y)| x <= y} is a relation on A.

Relations
(sections 5.1)

- Let A_1, A_2, … , A_n be sets. An n-ary relation on these sets (in this order) is a subset of $A_1 \times A_2 \times \ldots \times A_n$.
- Most of the times we consider $n = 2$.

Relations as Subsets

A: Siblinghood. A = {people}

Because relations are just subsets, all the usual set theoretic operations are defined between relations which belong to the same Cartesian product.

Q: Suppose we have relations on {1,2} given by $R = \{(1,1), (2,2)\}$, $S = \{(1,1),(1,2)\}$. Find:

1. The union $R \cup S$
2. The intersection $R \cap S$
3. The symmetric difference $R \oplus S$
4. The difference $R - S$
5. The complement of R

Relations as Subsets

A: $R = \{(1,1),(2,2)\}$, $S = \{(1,1),(1,2)\}$

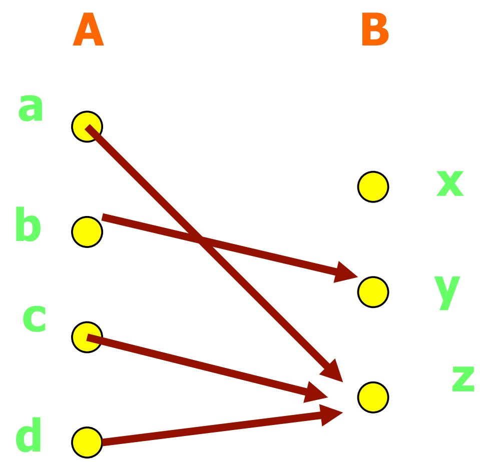
1. $R \cup S = \{(1,1),(1,2),(2,2)\}$
2. $R \cap S = \{(1,1)\}$
3. $R \oplus S = \{(1,2),(2,2)\}$.
4. $R - S = \{(2,2)\}$.
5. $\overline{R} = \{(1,2),(2,1)\}$

Functions (5.1,5.2,5.3)

- For non-empty sets A and B, a function (mapping) f from A to B (f: A → B) is a relation f (a subset of AxB) from A to B in which every element a ε A, the relation f contains exactly one pair of the form (a,b). The element (a,b) ε f is abbreviated as f(a) = b.
- A is the domain of f
  - B is the codomain of f
  - if f(a) = b, b is the image of a; a is the preimage of b
  - f is treated as a set
  - The range of f is the set {f(a): a ε A} = {b|(a,b) ε f}. The range is the set of all possible "output values" for f.

Example

- f(a) = z
- the image of d is z
- the domain of f is A = {a, b, c, d}
- the codomain is B = {x, y, z}
- f(A) = {y, z}
- the preimage of y is b
- the preimages of z are a, c and d
- f({c,d}) = {z}
- The range of f is {y,z}



Functions

- f: A→ B means
  - all a ε A have an image b ε B
  - some b ε B may not have a preimage a ε V
  - some b ε B may have more than one preimages a ε B
- f(A) denotes the subset X ⊆ B such that for any x ε X, there exists an element a ε A such that f(a) = x, and for any y ε B-X, there does not exist any a ε A such that f(a) = y.
  - X is called the range of f.

Functions

- Three things:
  - A function can be viewed as sending (mapping) elements from one set A to another set B.
  - Such a function can be regarded as a relation from A → B.
  - For every input value a (of A), there is exactly one output value f(a). (Vertical line test)

Some useful functions

- floor functions: real R → integer Z
  - floor(x) = greatest integer ≤ x

    $= \lfloor x \rfloor$
- ceiling functions: real R → integer Z
  - ceiling(x) = least integer ≥ x

    $= \lceil x \rceil$
- $\lfloor 3.5 \rfloor = 3$; $\lfloor \pi \rfloor = 3$; $\lfloor -3.5 \rfloor = -4$
- $\lceil 3.5 \rceil = 4$; $\lceil \pi \rceil = 4$; $\lceil -3.5 \rceil = -3$

Some examples

- g: Z x Z → Z which is defined as

  g((m,n)) = g(m,n) = 6m – 9n.

  – Note that g = {((m,n),6m-9n) | (m,n) ε Z x Z}

  – Domain is Z x Z

  – Codomain is Z

  – Range is { 3x : x ε Z} (why? Discussed in the class)

- A = {p,q,r,s}; B = {0,1,2};

  f = {(p,0), (q,1), (r,2), (s,2)}

  – f is a function with domain A, codomain B and range B.

Equality of functions

- Two functions $f:A \rightarrow B$ and $g:C \rightarrow D$ are equal if $A=C$, $B=D$ and $f(x)=g(x)$ for all $x \in A$.
- Caution: $f:Z \rightarrow N$ and $g:Z \rightarrow Z$, $f(x) = |x| +2$, and $g(x) = |x| +2$ are technically not equal.

Injective (one-to-one) Surjective (onto) functions

- A function f: A → B is:
  - Injective (one-to-one) if for every x,y ε A, x ≠ y, f(x) ≠ f(y). Equivalently:

    $$\forall x, y \in A, x \neq y \rightarrow f(x) \neq f(y)$$
    $$\text{Contraposition: } \forall x, y \in A, f(x) = f(y) \rightarrow x = y$$

  - Surjective (onto) if for every b ε B, there is an element a ε such that f(a) = b.

    $$\forall b \in B \; \exists a \in A \; f(a) = b$$

  - Bijective (one-to-one and onto) if f is injective and bijective.

one-to-one (injective) functions

- f: $Z^+ \rightarrow Z^+$ where $f(a) = a^2$ is one-to-one
- f: $Z \rightarrow Z^+$ where $f(a) = a^2$ is not one-to-one
- floor and ceiling function is not one-to-one.

onto (surjective) functions

- $f: Z \rightarrow Z$, $f(x) = x + 1$ is onto.
- $\lfloor . \rfloor: R \rightarrow Z$ is onto, but not one-to-one
- $\lceil . \rceil: R \rightarrow Z$ is onto, but not one-to-one

one-to-one and onto (bijection)

- f: R → R, f(x) = x + 1 is one-to-one correspondence
- f: [0,1] → [0,1/3], f(x) = x/3 is one-to-one and onto.
- f: R → R^+, f(x) = x^2 is not one-to-one, but onto.
- f: R → R, f(x) = x^3 is injective and surjective.

How to show a function f: A → B is injective?

- Direct approach: $\forall x, y \in A, x \neq y \rightarrow f(x) \neq f(y)$
  - Consider arbitrary x,y ε A; x ≠ y

    …. Reduction steps with the goal to show that f(x) ≠ f(y)
- Contraposition approach:

  $$\forall x, y \in A, f(x) = f(y) \rightarrow x = y$$

  - Suppose x,y ε A; f(x) = f(y)

    ….. Reduction steps with the goal to show that x=y.
- Often contrapositive approach is easy when f is an algebraic function

How to show a function f: A → B is surjective?

- f is surjective if for all b ε B, there exists a ε A such that f(a) = b. That is

  f is surjective if $\forall b \in B\ \exists a \in A\ f(a) = b$

- Contrapositive approach:

  not ($\forall b \in B\ \exists a \in A\ f(a) = b$ → f is not surjective

  i.e. $\exists b \in B\ \forall a \in A\ f(a) \neq b$ → f is not surjective

How to show a function f: A → B is surjective?

- f: R – {0} → R – {0}, f(x) = 1/x +1. Is it surjective?
  - Let b ε B be an arbitrary element. Let f(x) = b for some x ε A. This means that

    $$1/x + 1 = b$$

    i.e. x = 1/(b-1). Now x is not defined if b=1.

  Therefore f, as defined above, is not an onto function.
- Show that g : Z x Z → Z x Z, g(m,n)=(m+n,m+2n)

  is bijective. (Discussed in the class)