# Homework #2 Solutions

## Section 1.4

**1.** Let $x_i, 1 \le i \le 5$, denote the amounts given to the five children.

**(a)** The number of integer solutions of $x_1 + \ldots + x_5 = 10,\ 0 \le x_i,\ 1 \le i \le 5$, is $\binom{5+10-1}{10} = \binom{14}{10}$. Here $n = 5, r = 10$.

**(b)** Giving each child one dime results in the equation $x_1 + \ldots + x_5 = 5, 0 \le x_i, 1 \le i \le 5$. There are $\binom{5+5-1}{5} = \binom{9}{5}$ ways to distribute the remaining five dimes.

**(c)** Let $x_5$ denote the amount for the oldest child. The number of solutions to $x_1 + \ldots + x_5 = 10, 0 \le x_i, 1 \le i \le 4, 2 \le x_5$ is the number of solutions to $y_1 + \ldots + y_5 = 8, 0 \le y_i, 1 \le i \le 5$, which is $\binom{5+8-1}{8} = \binom{12}{8}$.

**8.** For the chocolate donuts there are $\binom{3+5-1}{5} = \binom{7}{5}$ distributions. There are $\binom{3+4-1}{4} = \binom{6}{4}$ ways to distribute the jelly donuts. By the rule of product there are $\binom{7}{5}\binom{6}{4}$ ways to distribute the donuts as specified.

**10.** Here we want the number of integer solutions for $x_1 + \ldots + x_6 = 100, x_i \ge 3, 1 \le i \le 6$. (For $1 \le i \le 6, x_i$ counts the number of times the face with $i$ dots is rolled.) This is equal to the number of nonnegative integer solutions there are to $y_1 + \ldots + y_6 = 82, y_i \ge 0, 1 \le i \le 6$. Consequently the answer is $\binom{6+82-1}{82} = \binom{87}{82}$.

**14. (a)** $\binom{8}{2,4,1,0,1}(3)^2(2)^4$

**(b)** The terms in the expansion have the form $v^a w^b x^c y^d z^e$ where $a, b, c, d, e$ are nonnegative integers that sum to 8. There are $\binom{5+8-1}{8} = \binom{12}{8}$ terms.

**19.** Here there are $r = 4$ nested **for** loops, so $1 \le m \le k \le j \le i \le 20$. We are making selections, with repetition, of size $r = 4$ from a collection of size $n = 20$. Hence the **print** statement is executed $\binom{20+4-1}{4} = \binom{23}{4}$ times.

**20.** Here there are $r = 3$ nested **for** loops and $1 \le m \le k \le j \le i \le 15$. So we are making selections, with repetition, of size $r = 3$ from a collection of size $n = 15$. Therefore the statement

$$\mathit{counter} := \mathit{counter} + 1$$

is executed $\binom{15+3-1}{3} = \binom{17}{3}$ times, and the final value of the variable *counter* is $10 + \binom{17}{3} = 690$.

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## Section 2.1

**4.** $(a) r \to q \quad (b) q \to p \quad (c) (s \wedge r) \to q$

**5.** (a) If triangle ABC is equilateral, then it is isosceles.
(b) If triangle ABC is not isosceles, then it is not equilateral.
(c) Triangle ABC is equilateral if and only if it is equiangular.
(d) Triangle ABC is isosceles but it is not equilateral.
(e) If triangle ABC is equiangular, then it is isosceles.

| $p$ | $q$ | $p \vee q$ | (a) $\neg(p \vee \neg q) \to \neg p$ | $p \to q$ | $q \to p$ | (d) $(p \to q) \to (q \to p)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

| $p$ | $q$ | $p \to q$ | $p \wedge (p \to q)$ | (e) $[p \wedge (p \to q)] \to q$ | (f) | (g) |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 0 |

**8.**

| $p$ | $q$ | $r$ | $q \to r$ | (b) $p \to (q \to r)$ | $p \to q$ | (c) $(p \to q) \to r$ | (h) |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

**10.**

| $p$ | $q$ | $r$ | $\overbrace{p \to (q \to r)}^{s}$ | $\overbrace{(p \to q) \to (p \to r)}^{t}$ | $s \to t$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |

**12. (a)** $[(p \wedge q) \wedge r] \to (s \vee t)$ is false (0) when $(p \wedge q) \wedge r$ is true (1) and $s \vee t$ is false (0). Hence $p, q$, and $r$ must be true (1) while $s$ and $t$ must be false (0).

**(b)** $[p \wedge (q \wedge r)] \rightarrow (s \veebar t)$ is false (0) when $p \wedge (q \wedge r)$ is true (1) and $s \veebar t$ is false (0). Hence $p, q$, and $r$ must be true (1) while $s$ and $t$ must have the same truth value.

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## Section 2.2

**2.**

| $p$ | $q$ | $p \wedge q$ | $p \vee (p \wedge q)$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 |

**4.** (1) $[(p \wedge q) \wedge r] \vee [(p \wedge q) \wedge \neg r] \Leftrightarrow (p \wedge q) \wedge (r \vee \neg r) \Leftrightarrow (p \wedge q) \wedge T_0 \Leftrightarrow p \wedge q$.
(2) $[(p \wedge q) \vee \neg q] \Leftrightarrow (p \vee \neg q) \wedge (q \vee \neg q) \Leftrightarrow (p \vee \neg q) \wedge T_0 \Leftrightarrow (p \vee \neg q)$.
Therefore, the given statement simplifies to $(p \vee \neg q) \rightarrow s$ or $(q \rightarrow p) \rightarrow s$.

**6.** (a) $\neg[p \wedge (q \vee r) \wedge (\neg p \vee \neg q \vee r)] \Leftrightarrow \neg p \vee (\neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r) \Leftrightarrow (\neg q \wedge \neg r) \vee [\neg p \vee (p \wedge q \wedge \neg r)] \Leftrightarrow (\neg q \wedge \neg r) \vee [T_0 \wedge (\neg p \vee (q \wedge \neg r))] \Leftrightarrow (\neg q \wedge \neg r) \vee [\neg p \vee (q \wedge \neg r)] \Leftrightarrow \neg p \vee [(\neg q \vee q) \wedge \neg r] \Leftrightarrow \neg p \vee \neg r$.
(b)$\neg[(p \wedge q) \rightarrow r] \Leftrightarrow \neg[\neg(p \wedge q) \vee r] \Leftrightarrow (p \wedge q) \wedge \neg r$.
(c)$p \wedge (q \vee \neg r)$
(d)$\neg p \wedge \neg q \wedge \neg r$

**14.** (a)

| $p$ | $q$ | $p \wedge q$ | $q \rightarrow (p \wedge q)$ | $p \rightarrow [q \rightarrow (p \wedge q)]$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 |

(b)Replace each occurrence of $p$ by $p \vee q$. Then we have the tautology $(p \vee q) \rightarrow [q \rightarrow [p \vee q) \wedge q]]$ by the first substitution rule. Since $(p \vee q) \wedge q \Leftrightarrow q$, by the absorption laws, it follows that $(p \vee q) \rightarrow [q \rightarrow q] \Leftrightarrow T_0$.

(c)

| $p$ | $q$ | $p \vee q$ | $p \wedge q$ | $q \rightarrow (p \wedge q)$ | $(p \vee q) \rightarrow [q \rightarrow (p \wedge q)]$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |

So the given statement is not a tautology. If we try to apply the second substitution rule to the result in part (a) we would replace the first occurrence of $p$ by $p \vee q$. But this does not result in a tautology because it is not a valid application of this substitution rule - for $p$ is not logically equivalent to $p \vee q$.

**16.** (a) $\neg p \Leftrightarrow (p \downarrow p)$
(b) $p \vee q \Leftrightarrow \neg\neg(p \vee q) \Leftrightarrow \neg(p \downarrow q) \Leftrightarrow (p \downarrow q) \downarrow (p \downarrow q)$
(c) $p \wedge q \Leftrightarrow \neg\neg p \wedge \neg\neg q \Leftrightarrow (\neg p \downarrow \neg q) \Leftrightarrow (p \downarrow p) \downarrow (q \downarrow q)$
(d) $p \rightarrow q \Leftrightarrow \neg p \vee q \Leftrightarrow (\neg p \downarrow q) \downarrow (\neg p \downarrow q) \Leftrightarrow [(p \downarrow p) \downarrow q] \downarrow [(p \downarrow p) \downarrow q]$
(e) $p \leftrightarrow q \Leftrightarrow (r \downarrow r) \downarrow (s \downarrow s)$ where $r$ stands for $[(p \downarrow p) \downarrow q] \downarrow [(p \downarrow p) \downarrow q]$ and $s$ for $[(q \downarrow q) \downarrow p] \downarrow [(q \downarrow q) \downarrow p]$

**18.**

| | |
|---|---|
| (a) $[(p \vee q) \wedge (p \vee \neg q)] \vee q$ | **Reasons** |
| $\Leftrightarrow [p \vee (q \wedge \neg q)] \vee q$ | Distributive Law of $\vee$ over $\wedge$ |
| $\Leftrightarrow (p \vee F_0) \vee q$ | $q \wedge \neg q \Leftrightarrow F_0$ (Inverse Law) |
| $\Leftrightarrow p \vee q$ | $p \vee F_0 \Leftrightarrow p$ (Identity Law) |

| | |
|---|---|
| (b) $(p \rightarrow q) \wedge [\neg q \wedge (r \vee \neg q)]$ | **Reasons** |
| $\Leftrightarrow (p \rightarrow q) \wedge \neg q$ | Absorbtion Law (and the Commutative Law of $\vee$) |
| $\Leftrightarrow (\neg p \vee q) \wedge \neg q$ | $p \rightarrow q \Leftrightarrow \neg p \vee q$ |
| $\Leftrightarrow \neg q \wedge (\neg p \vee q)$ | Commutative Law of $\wedge$ |
| $\Leftrightarrow (\neg q \wedge \neg p) \vee (\neg q \wedge q)$ | Distributive Law of $\wedge$ over $\vee$ |
| $\Leftrightarrow (\neg q \wedge \neg p) \vee F_0$ | Inverse Law |
| $\Leftrightarrow \neg q \wedge \neg p$ | Identity Law |
| $\Leftrightarrow \neg(q \vee p)$ | DeMorgan's Law |

**Problem:** Rephrase problem 17 (page 66) using only the *not, or, and* connectives.
(a) $\neg(\neg(p \vee q)) \Leftrightarrow \neg(\neg p \wedge \neg q)$
(b) $\neg(\neg(p \wedge q)) \Leftrightarrow \neg(\neg p \vee \neg q)$