Name: ———————————- **Student Id:** ——————— **Group:** ——

Midterm 2 (MACM101-D2)
March 10, 2014.
Test duration: 50 minutes
Total: 35 points
The bonus question is worth 5 points.

1. (8 points) Our universe is $\mathbb{Z}^+$.

   (a) Prove that the largest number you cannot write as the sum of 4 or 7 is 17?

   Ans: Let $S(n) : n \in \mathbb{Z}^+$ can be written as $n = 4a + 7b$ where $a, b \in \mathbb{N}$. Since there do not exist $a, b \in \mathbb{N}$ such that $17 = 4a + 7b$, $S(17)$ is false. Using the principle of strong induction we now show that $S(n)$ is true $\forall n \geq 18$.

   Basis: We can show that $S(18), S(19), S(20), S(21)$ are true.
   Induction hypothesis: Suppose for arbitrary $k \geq 21$, $S(18) \wedge S(19) \wedge S(20) \wedge \ldots \wedge S(k)$ is true.

   Now we need to show that $S(k+1)$ is true.
   We notice that $k + 1 - 4 \geq 18$ since $k \geq 21$. Thus $S(k+1-4)$ is true by the induction hypothesis. Therefore, $k + 1$ can be written as the sum of 4 or 7. By the principle of strong induction we can conclude that $S(n)$ is true $\forall n \geq 18$.
   Therefore, the largest number that cannot be written as the sum of 4 or 7 is 17.

   (b) Disprove that there exists a largest number you cannot write as the sum of 6 or 9 or 12.

   Ans: Note that 3 is a common factor (greatest common divisor) of 6, 9 and 12. Therefore, $6a + 9b + 12c$ is an integer divisible by 3 for any $a, b, c \in \mathbb{N}$. This means that any positive integer not divisible by 3 cannot be written as the sum of 6 9 or 12. The claim that there does not exist a largest finite number you cannot write as the sum of 6, 9 or 12 is proved by contradiction as follows. Suppose there is such a largest number α. If α is divisible by 3, $\alpha + 1 > \alpha$ and $\alpha + 1$ is not divisible by 3. Therefore, $\alpha + 1$ cannot be written as the sum of 6, 9 or 12 – a contradiction. If α is not divisible by 3, $\alpha + 3 > \alpha$ and $\alpha + 3$ is not divisible by 3. Therefore, $\alpha + 3$ cannot be written as the sum of 6, 9 or 12 – a contradiction.

2. (5 points) Prove by contradiction that $\sqrt{7}$ is an irrational number.

   Ans: The answer is very similar to that of question Part B, Q.1 of homework 4.

3. (7 points) A sequence of numbers $a_1, a_2, \ldots$ is defined by

$$a_1 = 1 \qquad a_2 = 1 \qquad a_n = a_{n-1} + a_{n-2},\ n \geq 3.$$

   (a) Determine the values of a_3, a_4, a_5, a_6 and a_7.
   Ans: $a_3 = a_2 + a_1 = 2$; $a_4 = 3$; $a_5 = 5$; $a_6 = 8$; $a_7 = 13$.

   (b) Prove that for all $n \geq 1, a_n < (\frac{1+\sqrt{5}}{2})^n$.
   Ans: Let $S(n) : a_n < (\frac{1+\sqrt{5}}{2})^n$. We will use the principle of strong induction.
   Basis: $S(1)$ is true since $a_1 = 1 < (\frac{1+\sqrt{5}}{2})$ since $\sqrt{5}$ is greater than 2. $S(2)$ is also true since $a_2 = 1 < (\frac{1+\sqrt{5}}{2})^2$.
   Induction hypothesis: Suppose $S(1) \wedge S(2) \wedge \ldots \wedge S(k)$ is true for an arbitrary $k \geq 2$.

   We now show that $S(k+1)$ is true, i.e. we need to show that $a_{k+1} < (\frac{1+\sqrt{5}}{2})^{k+1}$. We will use the direct proof method.

$$\begin{aligned}
a_{k+1} &= a_k + a_{k-1} \\
&< (\tfrac{1+\sqrt{5}}{2})^k + (\tfrac{1+\sqrt{5}}{2})^{k-1} \text{ by the induction hypothesis.} \\
&= (\tfrac{1+\sqrt{5}}{2})^{k-1}(1 + \tfrac{1+\sqrt{5}}{2}) \\
&< (\tfrac{(1+\sqrt{5}}{2})^{k-1}(\tfrac{1+\sqrt{5}}{2})^2 \text{ since } (1 + \tfrac{1+\sqrt{5}}{2}) < (\tfrac{1+\sqrt{5}}{2})^2 \\
&= (\tfrac{(1+\sqrt{5}}{2})^{k+1}.
\end{aligned}$$

   Thus $S(k+1)$ is true. Therefore, by the principle of strong induction, $\forall n \geq 1, S(n)$ is true.

4. (5 points) Using proof by cases, show that for any $n \in (\mathbb{N} - \{0, 1\})$,

   (a) $n = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil$.
   Ans: $n \in \mathbb{N}$ is either

   - **even :** $n = 2k, k \in \mathbb{N}$

     In this case we need to show that $2k = \lfloor \frac{2k}{2} \rfloor + \lceil \frac{2k}{2} \rceil = k + k$.

- **odd:** $n = 2k+1, k \in \mathbb{N}$

  In this case $2k+1 = \lfloor \frac{2k+1}{2} \rfloor + \lceil \frac{2k+1}{2} \rceil = k + k + 1$.

Thus the statement $n = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil$ is true $\forall n \in \mathbb{N}$.

(b) (Bonus) $\lceil \log_2 n \rceil = \lfloor \log_2 (n-1) \rfloor + 1$.

Ans: $n \in \mathbb{N} - \{0, 1\}$ is either

- n **is a perfect power of 2, i.e** $n = 2^k, k \in \mathbb{N}$

  In this case $\lceil \log_2 2^k \rceil = k$ and $\lfloor \log_2 (2^k - 1) \rfloor = k - 1$.

  Therefore, $\lceil \log_2 n \rceil = \lfloor \log_2 (n-1) \rfloor + 1$ is true for any $n \in \mathbb{N} - \{0, 1\}$ which is a perfect power of 2.

- n **is not a perfect power of 2, i.e** $2^{k-1} < n < 2^k$, **for some** $k \in \mathbb{N}$

  In this case $\lceil \log_2 n \rceil = k$; and $\lfloor \log_2 (n-1) \rfloor = k - 1$.
  Therefore, $\lceil \log_2 n \rceil = \lfloor \log_2 (n-1) \rfloor + 1$ is true any $n \in \mathbb{N} - \{0, 1\}$ which is not a perfect power of 2.

Thus the statement $\lceil \log_2 n \rceil = \lfloor \log_2 (n-1) \rfloor + 1$ is true $\forall n \in \mathbb{N} - \{0, 1\}$.

5. (5 points) Let A, B and C are subsets of the universal set U. Using the properties of union, intersection and complement and known set laws, simplify the following

   (a) $\overline{\overline{(A \cup B) \cap C} \cup \overline{B}}$.

   Ans:

$$\begin{aligned} \overline{\overline{(A \cup B) \cap C} \cup \overline{B}} &= \overline{\overline{(A \cup B) \cap C \cap B}} \text{ DeMorgan's Laws} \\ &= (A \cup B) \cap C \cap B \text{ Laws of Double Complement} \\ &= (A \cup B) \cap B \cap C \text{ Commutative Laws} \\ &= B \cap C \text{ Absorption Laws} \end{aligned}$$

6. (5 point) If two integers are selected at random and without replacement from $\{1, 2, 3, ..., 199, 200\}$, what is the probability the two selected integers are consecutive i.e. integers i and $i+1$ for any integer $i \in [1, 200)$?

   Ans: Here the sample space has $\binom{200}{2}$ elements. Each element of the sample space is a 2-tuple (i, j), $1 \leq i < j \leq 200$. Of these $\binom{200}{2}$ elements, 199 elements are of the type $(i, i+1), 1 \leq i \leq 199$. Therefore, the probability of selecting two consecutive integers is $\frac{199}{\binom{200}{2}}$ which is $\frac{1}{100}$.