**Name:** ——————————— **Student Id:** —————— **Group:** ——

**Midterm 2 (MACM101-D1)**
March 10, 2014.
Test duration: 50 minutes
Total: 35 points
The bonus question is worth 5 points.

1. (8 points) Our universe is $\mathbb{Z}^+$.

   (a) Prove that the largest number you cannot write as the sum of 5 or 9 is 31?

   **Ans:** Let $S(n) : n \in \mathbb{Z}^+$ can be written as $n = 5a + 9b$ where $a, b \in \mathbb{N}$. Since there do not exist $a, b \in \mathbb{N}$ such that $31 = 5a + 9b$, $S(31)$ is false. Using the principle of strong induction we now show that $S(n)$ is true $\forall n \geq 32$.

   **Basis:** We can show that $S(32), S(33), S(34), S(35), S(36)$ are true.
   **Induction hypothesis:** Suppose for arbitrary $k \geq 36$, $S(32) \wedge S(33) \wedge S(34) \wedge \ldots \wedge S(k)$ is true.

   Now we need to show that $S(k + 1)$ is true.

   We notice that $k+1-5 \geq 32$ since $k \geq 36$. Thus $S(k+1-5)$ is true by the induction hypothesis. Therefore, $k + 1$ can be written as the sum of 5 or 9. By the principle of strong induction we can conclude that $S(n)$ is true $\forall n \geq 32$.

   Therefore, the largest number that cannot be written as the sum of 5 or 9 is 31.

   (b) Disprove that there exists a largest number you cannot write as the sum of 6 or 8.

   **Ans:** Note that 2 is a common factor (greatest common divisor) of 6 and 8. Therefore, $6a + 8b$ is an even integer for any $a, b \in \mathbb{N}$. This means that any odd positive integer cannot be written as the sum of 6 or 8. The claim that there does not exist a largest finite number you cannot write as the sum of 6 or 8 is proved by contradiction as follows. Suppose there is such a largest number $\alpha$. If $\alpha$ is even, $\alpha + 1 > \alpha$ and $\alpha + 1$ is odd. Therefore, $\alpha + 1$ cannot be written as the sum of 6 or 8

– a contradiction. If $\alpha$ is odd, $\alpha + 2 > \alpha$ and $\alpha + 2$ is odd. Therefore, $\alpha + 2$ cannot be written as the sum of 6 or 8 – a contradiction.

2. (5 points) Prove by contradiction that $\sqrt{5}$ is an irrational number.

   **Ans:** The answer is very similar to that of question Part B, Q.1 of homework 4.

3. (7 points) A sequence of numbers $a_1, a_2, \ldots$ is defined by

$$a_1 = 1 \qquad a_2 = 2 \qquad a_n = a_{n-1} + a_{n-2},\ n \geq 3.$$

   (a) Determine the values of $a_3, a_4, a_5, a_6$ and $a_7$.

   **Ans:** $a_3 = a_2 + a_1 = 3$; $a_4 = 5$; $a_5 = 8$; $a_6 = 13$; $a_7 = 21$.

   (b) Prove that for all $n \geq 1, a_n < (\frac{7}{4})^n$.

   **Ans:** Let $S(n) : a_n < (\frac{7}{4})^n$. We will use the principle of strong induction.
   **Basis:** $S(1)$ is true since $a_1 = 1 < (\frac{7}{4})$. $S(2)$ is also true since $a_2 = 2 < (\frac{7}{4})^2$.
   **Induction hypothesis:** Suppose $S(1) \wedge S(2) \wedge \ldots \wedge S(k)$ is true for an arbitrary $k \geq 2$.

   We now show that $S(k+1)$ is true, i.e. we need to show that $a_{k+1} < (\frac{7}{4})^{k+1}$. We will use the direct proof method.

$$\begin{aligned}
a_{k+1} &= a_k + a_{k-1} \\
&< (\tfrac{7}{4})^k + (\tfrac{7}{4})^{k-1} \text{ by the induction hypothesis.} \\
&= (\tfrac{7}{4})^{k-1}(1 + \tfrac{7}{4}) \\
&< (\tfrac{7}{4})^{k-1}(\tfrac{7}{4})^2 \text{ since } (1 + \tfrac{7}{4}) < (\tfrac{7}{4})^2 \\
&= (\tfrac{7}{4})^{k+1}.
\end{aligned}$$

   Thus $S(k+1)$ is true. Therefore, by the principle of strong induction, $\forall n S(n)$ is true.

4. (5 points) Using proof by cases, show that for any $n \in \mathbb{N}^+ - \{1\}$,

   (a) $n = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil$.

   **Ans:** $n \in \mathbb{N}$ is either

   - **even :** $n = 2k, k \in \mathbb{N}$

     In this case we need to show that $2k = \lfloor \frac{2k}{2} \rfloor + \lceil \frac{2k}{2} \rceil = k + k$.
   - **odd:** $n = 2k + 1, k \in \mathbb{N}$

     In this case $2k + 1 = \lfloor \frac{2k+1}{2} \rfloor + \lceil \frac{2k+1}{2} \rceil = k + k + 1$.

Thus the statement $n = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil$ is true $\forall n \in \mathbb{N}$.

(b) (Bonus) $\lceil \log_2 n \rceil = \lfloor \log_2 (n-1) \rfloor + 1$.

**Ans:** $n \in \mathbb{N} - \{0, 1\}$ is either

- **$n$ is a perfect power of 2, i.e $n = 2^k, k \in \mathbb{N}$**

  In this case $\lceil \log_2 2^k \rceil = k$ and $\lfloor \log_2 (2^k - 1) \rfloor = k - 1$.

  Therefore, $\lceil \log_2 n \rceil = \lfloor \log_2 (n-1) \rfloor + 1$ is true for any $n \in \mathbb{N} - \{0, 1\}$ which is a perfect power of 2.
- **$n$ is not a perfect power of 2, i.e $2^{k-1} < n < 2^k$, for some $k \in \mathbb{N}$**

  In this case $\lceil \log_2 n \rceil = k$; and $\lfloor \log_2 (n-1) \rfloor = k - 1$.
  Therefore, $\lceil \log_2 n \rceil = \lfloor \log_2 (n-1) \rfloor + 1$ is true any $n \in \mathbb{N} - \{0, 1\}$ which is not a perfect power of 2.

Thus the statement $\lceil \log_2 n \rceil = \lfloor \log_2 (n-1) \rfloor + 1$ is true $\forall n \in \mathbb{N} - \{0, 1\}$.

5. (5 points) Let $A, B$ and $C$ are subsets of the universal set $U$. Using the properties of union, intersection and complement and known set laws, simplify the following

(a) $\overline{\overline{(A \cup B) \cap C} \cup \overline{B}}$.

**Ans:**

$$\begin{aligned}
\overline{\overline{(A \cup B) \cap C} \cup \overline{B}} &= \overline{\overline{(A \cup B) \cap C \cap B}} \text{ DeMorgan's Laws} \\
&= (A \cup B) \cap C \cap B \text{ Laws of Double Complement} \\
&= (A \cup B) \cap B \cap C \text{ Commutative Laws} \\
&= B \cap C \text{ Absorption Laws}
\end{aligned}$$

6. (5 point) If two integers are selected at random and without replacement from $\{1, 2, 3, ..., 99, 100\}$, what is the probability the two selected integers are consecutive?

**Ans:** Here the sample space has $\binom{100}{2}$ elements. Each element of the sample space is a 2-tuple $(i, j)$, $1 \leq i < j \leq 100$. Of these $\binom{100}{2}$ elements, 99 elements are of the type $(i, i+1), 1 \leq i \leq 99$. Therefore, the probability of selecting two consecutive integers is $\frac{99}{\binom{100}{2}}$ which is $\frac{1}{50}$.