# Placing balls in bins

# Problem of balls in bins

- Count the number of ways to place a collection A of $m \geq 1$ balls into a collection B of n bins, $n \geq 1$.
- Balls are labeled (distinguishable) or unlabeled (indistinguishable)
- Bins are labeled (distinguishable) or unlabeled (indistinguishable)
- Placement is either unrestricted, injective (one-to-one) or surjective (onto)
- We thus have 12 cases. However, we will ignore the situation  when both the balls and the bins are unlabeled. Thus, effectively we will consider  the other 9 case.

## What do we know from our earlier studies?

- The number of integer solutions to $x_1+x_2+ \ldots\ldots + x_n = m$ when
  - $\forall\ 1 \leq i \leq n,\ \ x_i \geq 0$ : **ans:** $\binom{n+m-1}{n-1}$
  - $\forall\ 1 \leq i \leq n,\ \ x_i \geq 1$ : **ans:** $\binom{m-1}{n-1}$
- C(n+m-1,n-1) is also the number of combinations of selecting m elements with repetitions from a set of n objects.
- This is the same problems as the problem of distributing m pennies to n kids.

## What do we know from our earlier studies?

- The four types of functions f: A → B where |A|=m & |B|=n:

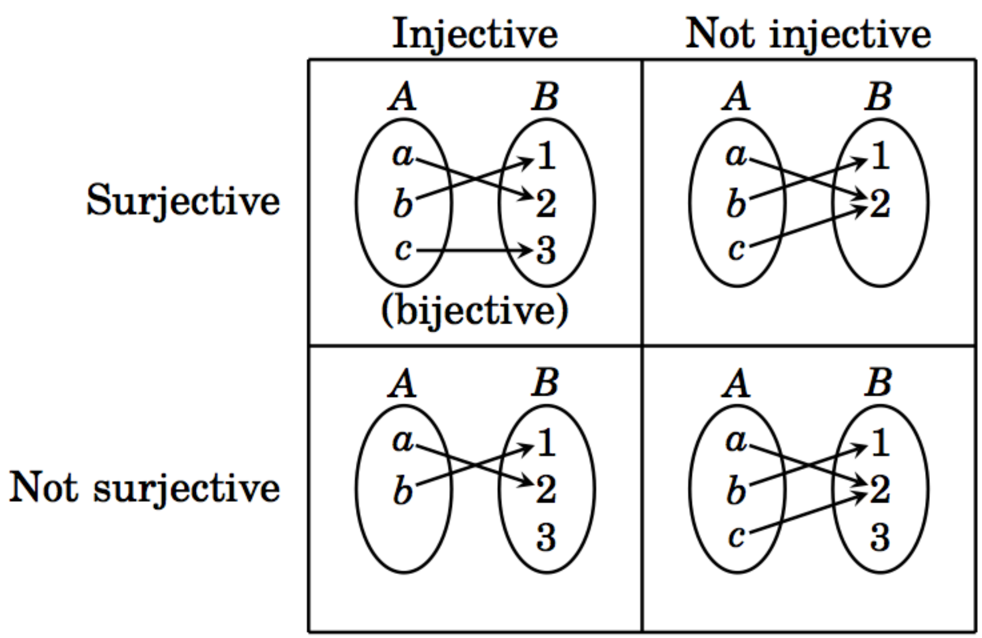


Figure 1

# 9 cases

# A: m labeled balls; B: n labeled bins; Unrestricted placements

- Interpretations:
  - Count the number of functions f: A → B.
  - We get one-to-one matching of ball plcements and function f by placing ball i into bin j if f(i)=j.
- Formula
  - Let g(m,n) denote the number of placements. Then $g(m,n)=n^m$.
  - When n=2, $g(m,n)=2^m$. This number is the same as the number of binary strings of length m.

# A: m labeled balls; B: n labeled bins; one-to-one placements

- Interpretations:
  - Count the number of injective functions f: A → B.
- Formula
  - Let g(m,n) denote the number of placements. Then g(m,n)=n(n-1)(n-2)….(n-m+1) = P(m,n).
  - When m > n, g(m,n)=0.
- Additional comments:
  - g(m,n)=n! when |A|=|B|. This function f on A is called a permutation.
  - The Pigeonhole Principle states that there is no injection if m > n: for any function in such a case, there must be at least one bin (pigeonhole) with at least two balls (pigeons).

# A: m labeled balls; B: n labeled bins; onto placements

- Interpretations:
  - Count the number of surjective functions f: A → B.
- Formula
  - Let $\hat{S}(m,n)$ denote the number of placements. Then

$$\hat{S}(m,n) \;=\; n^m - \binom{n}{n-1}\hat{S}(m,n-1) - \binom{n}{n-2}\hat{S}(m,n-2) - \ldots - \binom{n}{1}\hat{S}(m,1).$$

- Additional comments:
  - Why does this work? $n^m$ is the number of functions. We then remove the number of onto functions whose range has n-1 elements; n-2 elements; etc.

# A: m unlabeled balls; B: n labeled bins; unrestricted placements

- Interpretations:
  - The number of integer solutions to $x_1 + \ldots + x_n = m$, $x_i \geq 0$.
  - Distributing n pennies to m kids, a kid may get 0 penny
- Let g(m,n) denote the number of placements. Then

$$g(m,n) = \binom{m+n-1}{n-1}$$

# A: m unlabeled balls; B: n labeled bins; one-to-one placements

- Interpretations:
  - Counts the number of subsets of {1,2, …, n} of size m.
- Let g(m,n) denote the number of placements. Then

  g(m,n) = $\binom{n}{m}$ or $C(n,m)$

# A: m unlabeled balls; B: n labeled bins; onto placements

- Interpretations:
  - The number of integer solutions to $x_1 + \ldots + x_n = m$, $x_i \geq 1$.
  - Distributing n pennies to m kids, a kid may get at least 1 penny
  - Counts the number of ways of writing m as a sum of n positive integers where different orderings are counted as different.
- Let g(m,n) denote the number of placements. Then g(m,n) = $\binom{m-1}{m-n}$ or $\binom{m-1}{n-1}$.

# A: m labeled balls; B: n unlabeled bins; unrestricted placements

- Interpretations:

Consider the four functions defined in Figure 1. The functions can be written as follows:

**Surjective – Injective:** $f = \{(a, 2), (b, 1), (c, 3)\}$

**Surjective – Not injective:** $f = \{(a, 2), (b, 1), (c, 2)\}$

**Not surjective – Injective:** $f = \{(a, 2), (b, 1)\}$

**Not surjective – Not injective:** $f = \{(a, 2), (b, 1), (c, 2)\}$

Consider another set of functions $f : A \rightarrow B$. Draw a figure similiar to Figure 1.

**Surjective – Injective:** $f = \{(a, 1), (b, 2), (c, 3)\}$

**Surjective – Not injective:** $f = \{(a, 1), (b, 2), (c, 1)\}$

**Not surjective – Injective:** $f = \{(a, 1), (b, 2)\}$

**Not surjective – Not injective:** $f = \{(a, 1), (b, 2), (c, 1)\}$

# A: m labeled balls; B: n unlabeled bins; unrestricted placements (contd)

- Interpretations (continued):

Note that the two sets of functions are the same when the bins are not labeled. When the bins are unlabeled, what we get is a partition of m elements as follows (for both the cases):

**Surjective – Injective:** The sets are $\{\{a\}, \{b\}, \{c\}\}$

**Surjective – Not injective:** The sets are $\{\{b\}, \{a, b\}\}$

**Not surjective – Injective:** The sets are $\{\{a\}, \{b\}\}$

**Not surjective – Not injective:** The sets are $\{\{b\}, \{a, c\}\}$

However, these two sets of functions are different if the bins are considered labeled.

# A: m labeled balls; B: n unlabeled bins; unrestricted placements (contd)

- Formula:

We have used $\hat{S}(m,n)$ to indicate the number of onto functions from $A$ to $B$. Therefore, the number of partitions of $m$ elements nto exactly $n$ blocks is $\frac{\hat{S}(m,n)}{n!}$. In the text $S(m,n)$ is used to indicate the number $\frac{\hat{S}(m,n)}{n!}$. $S(m,n)$ is thus used in the text to indicate the number of partitions of $\{1,2,...,n\}$ into at most $n$ blocks, and $S(m,n)$ is called Stirling numbers of the second kind.

Therefore, the number of partitions of $A$ into at most $n$ blocks is

$$\frac{\hat{S}(m,1)}{1!} + \frac{\hat{S}(m,2)}{2!} + \ldots + \frac{\hat{S}(m,n)}{n!}, \text{ or}$$
$$S(m,1) + S(m,2) + ... + S(m,n).$$

# A: m labeled balls; B: n unlabeled bins; one-to-one placements (contd)

- Interpretation: Either you can do it (when $m \leq n$) or you cannot do it  when $m > n$.
- Count = 1 when $m \leq n$, otherwise it is zero

# A: m labeled balls; B: n unlabeled bins; onto placements

- Interpretations:
  - Counts the number of partitions of {1,2, …, m} into exactly n nonempty blocks.
- Formula: S(m,n) = $\frac{\hat{S}(m,n)}{n!}$