Homework #1: MACM-300

Reading: Sipser; Chapter 0

Distributed on Jan 9; due on Jan 16 (in class)

Anoop Sarkar – anoop@cs.sfu.ca

Only submit answers for questions marked with †.

(1) A relation R on set S is *irreflexive* if aRa is false for all a in S and *asymmetric* if aRb implies bRa is false. Show that any asymmetric relation must be irreflexive.

(2) Show that $P \rightarrow Q$ is equivalent to $\neg P \vee Q$

(3) Prove by construction that for each even number n greater than 2, there exists a 3-regular graph with n nodes (see Sipser, p.21). Draw a 3-regular graph for $n = 4$.

(4) Sipser, q0.10

(5) Prove by induction that

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

(6) † A palindrome can be defined as (Defn 1) a string that reads the same forward and backward (e.g. *malayalam*) or alternatively using the following definition (Defn 2):

1. ϵ is a palindrome (ϵ is the empty string)
2. If a is any symbol, then the string a is a palindrome
3. If a is any symbol and x is a palindrome, then axa is a palindrome
4. Nothing is a palindrome unless it follows from (1) through (3)

Prove by induction that the two definitions are equivalent. In order to show that the two definitions are equivalent we need to show it in two directions. Defn 1 implies Defn 2 ($\Rightarrow$) and Defn 2 implies Defn 1 ($\Leftarrow$).

(7) † Sipser, q0.11

(8) † Consider any set of strings L over the alphabet $\Sigma = \{a, b, c, \ldots, z\}$. Binary relation R is defined as follows: uRw is TRUE iff strings u and w begin with same symbol from Σ.

a. Show that R defines an **equivalence relation** over the strings in set L.

b. A **partition** of a set S is the collection of disjoint subsets of S whose union is S. Show that R induces a partition of L.

c. Provide an example of L and the partition of set L based on R.

(9) † Let $\mathbf{P}$ be a set of properties of relations over set S. The $\mathbf{P}$-*closure* of a relation R is the smallest relation R' that includes all pairs of R and possesses the properties in $\mathbf{P}$.
For example, the *transitive closure* of R, denoted by R^+, is defined by:

1. If (a, b) is in R, then (a, b) is in R^+
2. If (a, b) is in R^+ and (b, c) is in R, then (a, c) is in R^+
3. Nothing is in R^+ unless it follows from (1) and (2)

The *reflexive and transitive closure* of R, denoted by R^* is defined by $R^+ \cup \{(a, a) : a \in S\}$.
Provide the *transitive closure*, the *reflexive and transitive closure* and the *symmetric closure* of the relation $R = \{(1, 2), (2, 3), (3, 4), (5, 4)\}$ on the set $S = \{1, 2, 3, 4, 5\}$.

(10) † *Equal sets* have the same elements, while *equivalent sets* have the same number of members. For *infinite* sets we show equivalence between two sets by providing a 1-1 mapping between elements from the two sets. A set is countably infinite iff:

1. we can establish a 1-1 mapping between the set and the set of natural numbers.
2. it is equivalent to (at least one) proper subset of itself.

Show that the set S of all strings over an alphabet Σ with one element is countably infinite using both methods.