MACM 300
Formal Languages and Automata

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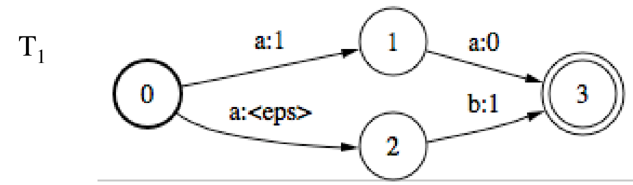
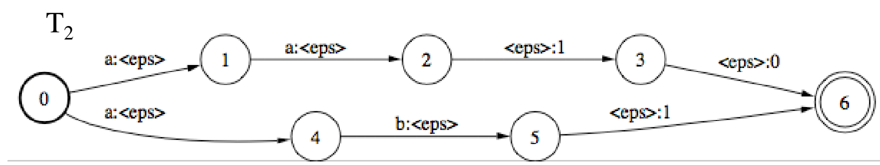
Finite-state transducers



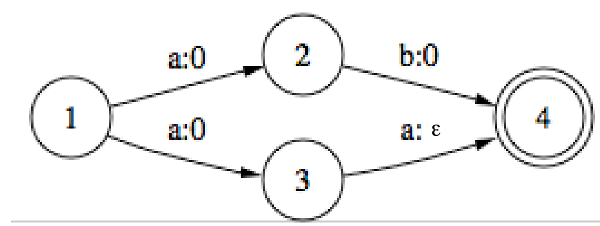
$R(T_1) = R(T_2) = \{ (aa, 10), (ab, 1) \}$

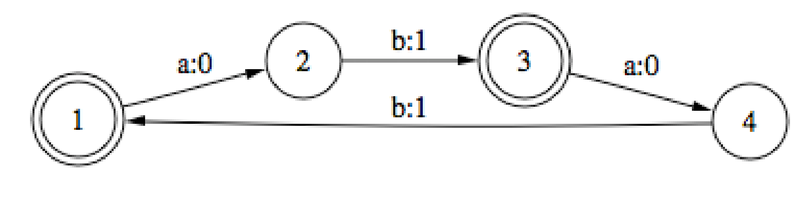


Finite-state transducers

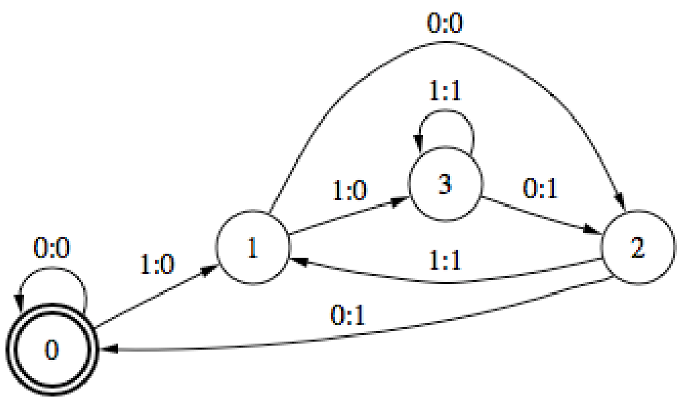
- a:0 is a notation for a mapping between two alphabets $a \in \Sigma_1$ and $0 \in \Sigma_2$
- Finite-state transducers (FSTs) accept pairs of strings
- Finite-state automata equate to regular languages and FSTs equate to regular relations
- e.g. $L = \{ (x^n, y^n) \mid n > 0, x \in \Sigma_1 \text{ and } y \in \Sigma_2\}$ is a regular relation accepted by some FST. It maps a string of x's into an equal length string of y's

Finite-state transducers





Finite-state transducers



Finite-state transducers

- Formal definition:
  - Q: finite set of states, $q_0, q_1, \ldots, q_n$
  - Σ: alphabet composed of input/output pairs i:o where $i \in \Sigma_1$ and $o \in \Sigma_2$ and so $\Sigma \subseteq \Sigma_1 \times \Sigma_2$
  - q_0: start state
  - F: set of final states
  - $\delta(q, i{:}o)$ is the transition function which returns a set of states

Regular relations

- A generalization of regular languages
- The set of regular relations is:
  - The empty set and (x,y) for all $x, y \in \Sigma_1 \times \Sigma_2$ is a regular relation
  - If R_1, R_2 and R are regular relations then:

    $$R_1 \cdot R_2 = \{(x_1x_2, y_1y_2) \mid (x_1,y_1) \in R_1, (x_2,y_2) \in R_2\}$$

    $$R_1 \cup R_2$$

    $$R^* = \cup_{i=0}^{\infty} R_i$$
  - There are no other regular relations

Finite-state transducers: Examples

- (a^n, b^n): map n a's into n b's
- rot13 encryption (the Caesar cipher): assuming 26 letters each letter is mapped to the letter 13 steps ahead (mod 26), e.g. *cipher* → *pvcure*
- reversal of a fixed set of words
- reversal of all strings upto fixed length k
- input: binary number n, and output: binary number $n+1$
- upcase or lowercase a string of any length
- *Pig latin: *pig latin is goofy* → *igpay atinlay is oofygay*
- *convert numbers into pronunciations,

  e.g. 230.34 two hundred and thirty point three four

Finite-state transducers

- Following relations are cannot be expressed as a FST
  - ($a^n b^n$, c^n): because $a^n b^n$ is not regular
  - reversal of strings of any length
  - $a^i b^j \rightarrow b^j a^i$ for any i, j
- Unlike regular languages, regular relations are not closed under intersection
  - ($a^n b^*$, c^n) ∩ ($a^* b^n$, c^n) produces ($a^n b^n$, c^n)
  - However, regular relations with input and output of equal lengths **are** closed under intersection

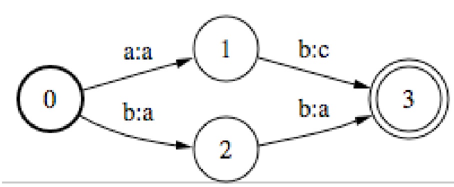
Regular Relations Closure Properties

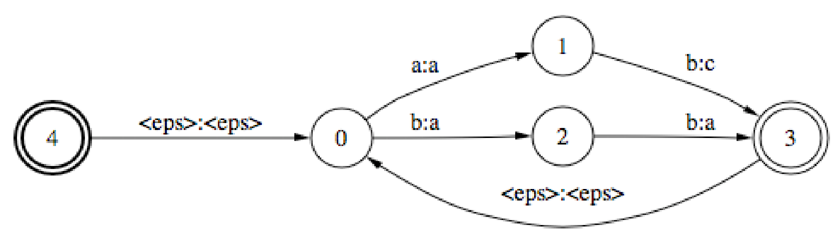
- New operations for regular relations:
  - *composition*
  - *project* input (or output) language to regular language; for FST t, input language = $\pi_1(t)$, output = $\pi_2(t)$
  - take a regular language and create the *identity* regular relation; for FSM f, let FST for identity relation be Id(f)
  - take two regular languages and create the *cross product* relation; for FSMs f & g, FST for cross product is $f \times g$
  - take two regular languages, and *mark each time the first language matches any string in the second language*

Regular Relations Closure Properties

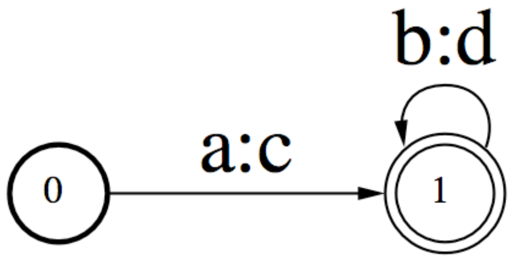
- Regular relations (rr) are ***closed*** under some operations
- For example, if R_1, R_2 are regular relns:
  - union ($R_1 \cup R_2$ results in R_3 which is a rr)
  - concatenation
  - iteration (R_1+ = one or more repeats of R_1)
  - Kleene closure (R_1* = zero or more repeats of R_1)
- However, unlike regular languages, regular relns are not closed under:
  - intersection (possible for equal length regular relns)
  - complement

Regular Relation/FST Kleene Closure

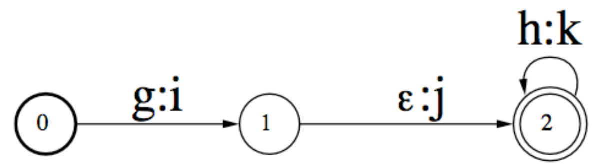




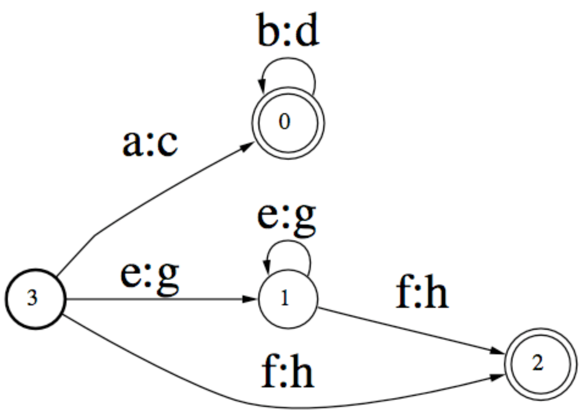
Regular Expressions for FSTs



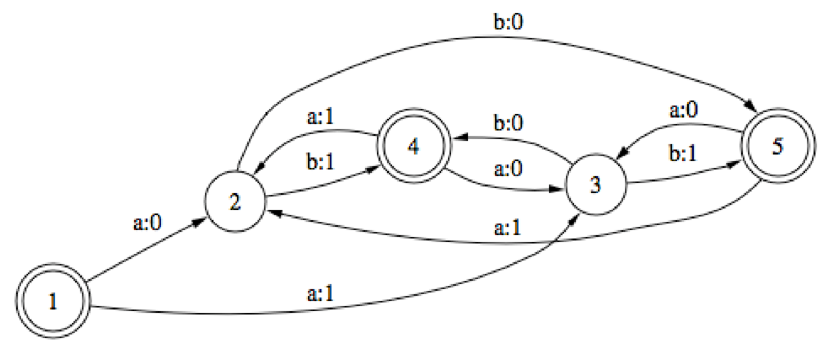
(a:c) (b:d)*



g:i ε:j (h:k)*

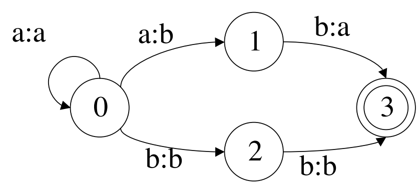


(a:c (b:d)*) ∪ ((e:g)* f:h)

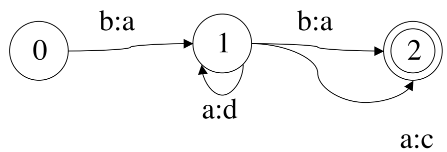


((a:0 ∪ a:1) (b:0 ∪ b:1))*

FSTs: Closure under Composition

T1:


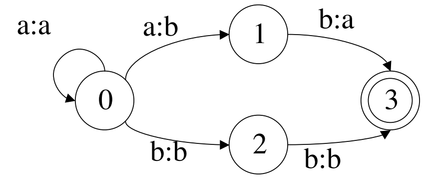
a^n ab := a^n ba
a^n bb := a^n bb

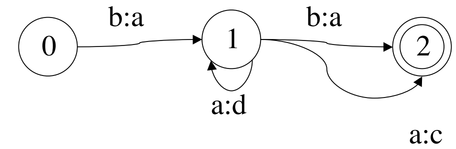
T2:


b a^n b := a d^n a
b a^n a := a d^n c

What is T_1 composed with T_2, written as T_1 o T_2 ?

Composing FSTs





0→1 a:b
0→2 b:b
2→3 b:b

0→1 b:a
1→2 b:a

0→0 a:a
1→3 b:a

1→1 a:d
1→2 a:c

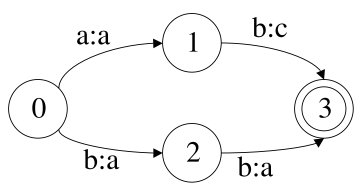
(0,0)→(1,1) a:a
(0,1)→(1,2) a:a
(2,0)→(3,1) b:a

(0,0)→(2,1) b:a
(0,1)→(2,2) b:a
(2,1)→(3,2) b:a

(0,1)→(0,1) a:d
(0,1)→(0,2) a:c

(1,1)→(3,1) b:d
(1,1)→(3,2) b:c

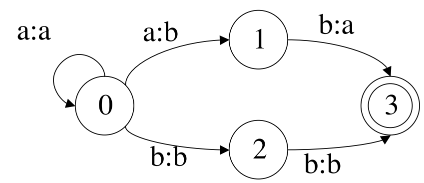
Composing FSTs

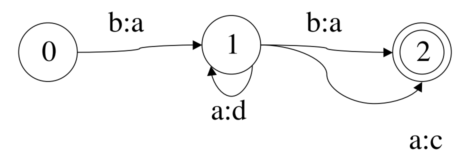
T_1 o T_2:


ab := ac
bb := aa

- The basic idea is similar to the closure of regular languages under union or intersection
- **But**, instead of cross-product of *states*, we consider cross-product of *edges*: compose those edges where output/input matches

Composing FSTs





0→1 a:b
0→2 b:b
2→3 b:b

0→1 b:a
1→2 b:a

0→0 a:a
1→3 b:a

1→1 a:d
1→2 a:c

(0,0)→(1,1) a:a
(0,1)→(1,2) a:a
(2,0)→(3,1) b:a

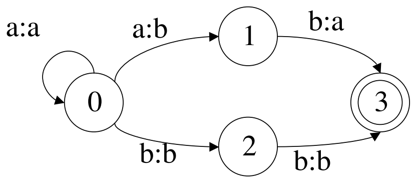
(0,0)→(2,1) b:a
(0,1)→(2,2) b:a
(2,1)→(3,2) b:a

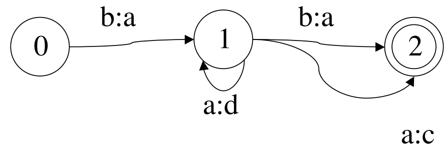
(0,1)→(0,1) a:d
(0,1)→(0,2) a:c

(1,1)→(3,1) b:d
(1,1)→(3,2) b:c

start with pair of final states

Composing FSTs





0→1 a:b
0→2 b:b
2→3 b:b

0→1 b:a
1→2 b:a

0→0 a:a
1→3 b:a

1→1 a:d
1→2 a:c

(0,0)→(1,1) a:a
(0,1)→(1,2) a:a
(2,0)→(3,1) b:a

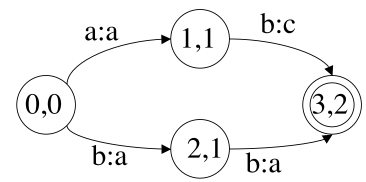
(0,0)→(2,1) b:a
(0,1)→(2,2) b:a
(2,1)→(3,2) b:a

(0,1)→(0,1) a:d
(0,1)→(0,2) a:c

(1,1)→(3,1) b:d
(1,1)→(3,2) b:c

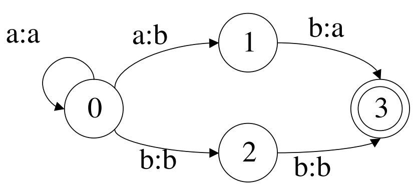
Composing FSTs

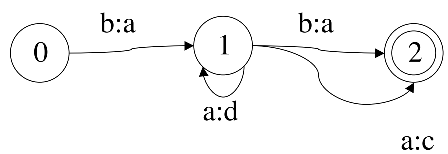
$T_1 \circ T_2$:



ab := ac
bb := aa

Composing FSTs





0→1 a:b
0→2 b:b
2→3 b:b

0→1 b:a
1→2 b:a

0→0 a:a
1→3 b:a

1→1 a:d
1→2 a:c

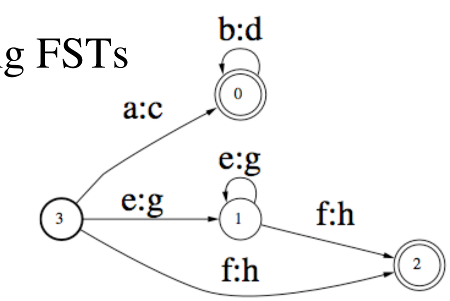
(0,0)→(1,1) a:a
(0,1)→(1,2) a:a
(2,0)→(3,1) b:a

(0,0)→(2,1) b:a
(0,1)→(2,2) b:a
(2,1)→(3,2) b:a

(0,1)→(0,1) a:d
(0,1)→(0,2) a:c

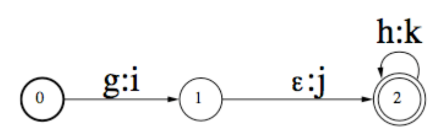
(1,1)→(3,1) b:d
(1,1)→(3,2) b:c

Composing FSTs

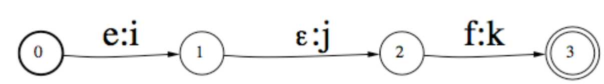


(a:c (b:d)*) ∪ ((e:g)* f:h)

g:i ε:j (h:k)*



e:i ε:j f:k



FST Composition

- Input: transducer S and T
- Transducer composition results in a new transducer with states and transitions defined by matching compatible input-output pairs.
- match(s,t) : defines the edges for the new composed FST, s is a state in S, t is a state in T

Cross-product FST

- For regular languages L_1 and L_2, we have two FSAs, M_1 and M_2

$$M_1 = (\Sigma_1, Q_1, q_1, F_1, \delta_1)$$
$$M_2 = (\Sigma_2, Q_2, q_2, F_2, \delta_2)$$

- Then a transducer accepting $L_1 \times L_2$ is defined as:

$$T = (\Sigma_1, \Sigma_2, Q_1 \times Q_2, \langle q_1, q_2 \rangle, F_1 \times F_2, \delta)$$
$$\delta(\langle s_1, s_2 \rangle, a, b) = \delta_1(s_1, a) \times \delta_2(s_2, b)$$
$$\text{for any } s_1 \in Q_1, s_2 \in Q_2 \text{ and } a, b \in \Sigma \cup \{\varepsilon\}$$

FST Composition

- match(s,t) =

  $\{ (s,t) \rightarrow^{x:z} (s',t') \mid s \rightarrow^{x:y} s' \in S.\text{edges and } t \rightarrow^{y:z} t' \in T.\text{edges} \} \cup$

  $\{ (s,t) \rightarrow^{x:\varepsilon} (s',t) \mid s \rightarrow^{x:\varepsilon} s' \in S.\text{edges} \} \cup$

  $\{ (s,t) \rightarrow^{\varepsilon:z} (s,t') \mid t \rightarrow^{\varepsilon:z} t' \in T.\text{edges} \}$
- **Correctness**: any path in composed transducer mapping *u* to *w* arises from a path mapping *u* to *v* in S and path mapping *v* to *w* in T, for some *v*

Summary

- Finite state transducers specify regular relations
  - Encoding computation as finite-state transducers
- Extension of regular expressions to the case of regular relations/FSTs
- FST closure properties: union, concatenation, composition
- FST special operations:
  - creating regular relations from regular languages (Id, cross-product);
  - creating regular languages from regular relations (projection)