CMPT-379

Compilers

Anoop Sarkar

http://www.cs.sfu.ca/~anoop

Programming Languages and Formal Language Theory

- We ask the question: *Does a particular formal language describe some key aspect of a programming language*
- Then we find out if that language **isn't** in a particular language class

Programming Languages and Formal Language Theory

- For example, if we abstract some aspect of the programming language structure to the formal language: $\{ww^R \mid \text{ where } w \in \{a, b\}^*, w^R \text{ is the reverse of } w\}$ we can then ask if this language is a regular language
- If this is false, i.e. the language is not regular, then we have to go beyond regular languages

Recursion in Regular Languages

- Consider a regular expression for arithmetic expressions:
  $2 + 3 * 4$
  $8 * 10 + -24$
  $2 + 3 * -2 + 8 + 10$

  ```
  ^\s*-?\s*\d+\s*((\+|\*)\s*-?\s*\d+\s*)*$
  ```

- *Can we compute the* meaning *of these expressions?*

Recursion in Regular Languages

- Construct the finite state automata and associate the meaning with the state sequence
- However, this solution is missing something crucial about arithmetic expressions – *what is it?*

Do Programming Languages belong to Regular Languages

- Consider the following arithmetic expressions
  - $(((2) + (3)) * (4))$
  - $((8) * ((10) + (-24)))$
- Map ($\rightarrow a$ and) $\rightarrow b$. Map everything else to ϵ (keep only the tree structure)
- This results in strings like $aaababbabb$ and $aabaababbb$
- What is a good description of this language? Let's call it L

Pumping Lemma proofs

- Is L a regular language?
- To show something is *not* a regular language, we use the **pumping lemma**
- For any infinite set of strings generated by a finite-state machine if you consider a string that is long enough from this set, there has to be a loop which visits the same state at least twice (from *the pigeonhole principle*)
- Thus, in a regular language L, there are strings x, y, z such that $xy^iz \in L$ for $i \geq 0$ where $y \neq \epsilon$

Pumping Lemma

- Pumping Lemma formal statement:
  - For all A that is in the set of regular languages,
  - there exists a p (p is called the pumping length)
  - such that for all $s \in A, |s| \geq p$,
  - there exists strings x, y, z such that $s = xyz$ and $|y| > 0$ and $|xy| \leq p$,
  - such that for all $i \geq 0$, $xy^iz \in A$.
- Try it on regular languages: $L(ab^*a)$ and $L((aa)^*)$. Construct minimal DFA for each one to find the value of p that is appropriate.

Pumping Lemma proofs

- Let L' be the intersection of L with the language L_1 defined by the regular expression a^*b^*
- Intersect the set $L = \{\epsilon, ab, abab, aabb, \ldots\}$ with $L_1 = \{\epsilon, a, b, aa, ab, aab, abb, bb, \ldots\}$
- Recall that RLs are closed under intersection, so L' must also be a RL. In fact, we can describe L' as the language $a^n b^n$ for $n \geq 0$

Pumping Lemma proofs

- For any choice of y (consider a^i or $a^i b$ or b^i) if we multiply y^n for $n \geq 0$ we get strings that are not in L'
- For example, for a string $aaabbb$ if we pick $y = ab$ and pick $n = 2$ we get a string $aaababbb$ which is not in L'
- Hence, the pumping lemma leads to the conclusion that L' is **not** regular
- This implies that L is not regular since RLs are closed under intersection
- What lies beyond the set of regular languages?

The Chomsky Hierarchy

- **unrestricted** or **type-0** grammars, generate the *recursively enumerable* languages, automata equals *Turing machines*
- **context-sensitive** or **type-1** grammars, generate the *context-sensitive* languages, automata equals *Linear Bounded Automata*
- **context-free** or **type-2** grammars, generate the *context-free* languages, automata equals *Pushdown Automata*
- **regular** or **type-3** grammars, generate the *regular* languages, automata equals *Finite-State Automata*

The Chomsky Hierarchy
A system of grammars $G = (N, T, P, S)$

- T is a set of symbols called terminal symbols.
  Also called the alphabet Σ
- N is a set of non-terminals, where $N \cap T = \emptyset$
  Some notation: $\alpha, \beta, \gamma \in (N \cup T)^*$
  N is sometimes called the set of variables V
- P is a set of production rules that provide a finite description of an infinite set of strings (a language)
- S is the start non-terminal symbol (similar to the start state in a FSA)

Languages

- Language defined by G: $L(G)$
  - $L(G)$: set of strings $w \in T^*$ derived from S
  - $S \Rightarrow^+ w$ (derives in 1 or more steps using rules in P)
  - w is a sentence of G
  - Sentential form: $S \Rightarrow^+ \alpha$ and α contains a mix of terminals and non-terminals
- Two grammars G_1 and G_2 are equivalent if $L(G_1) = L(G_2)$

The Chomsky Hierarchy: $G = (N, T, P, S)$ where, $\alpha, \beta, \gamma \in (N \cup T)^*$

- **unrestricted** or **type-0** grammars: $\alpha \rightarrow \gamma$, such that $\alpha \neq \epsilon$
- **context-sensitive** or **type-1** grammars: $\alpha \rightarrow \gamma$, where $|\gamma| \geq |\alpha|$
  CSG Normal Form: $\alpha A \beta \rightarrow \alpha\gamma\beta$, such that $\gamma \neq \epsilon$ and $S \rightarrow \epsilon$ if $\epsilon \in L(G)$
- **context-free** or **type-2** grammars: $A \rightarrow \gamma$
- **regular** or **type-3** grammars: $A \rightarrow a\ B$ or $A \rightarrow a$

Regular grammars: **right-linear CFG**: $L(G) = L(a^*b^*)$

$$
\begin{aligned}
A &\rightarrow a\ A &\quad (1)\\
A &\rightarrow \epsilon &\quad (2)\\
A &\rightarrow b\ B &\quad (3)\\
B &\rightarrow b\ B &\quad (4)\\
B &\rightarrow \epsilon &\quad (5)
\end{aligned}
$$

- Input: bb
- Derivation using sentential forms: $A \Rightarrow bB \Rightarrow bbB \Rightarrow bb\epsilon = bb$

Context-free grammars: $L(G) = \{a^n b^n \mid n \geq 0\}$

$$
\begin{aligned}
S &\rightarrow a\ S\ b\\
S &\rightarrow \epsilon
\end{aligned}
$$

- Input: $aabb$
- Derivation using sentential forms:
  $S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aa\epsilon bb = aabb$

Context-free grammars: $L(G) = \{a^n \mid n \geq 0\}$

$$
\begin{aligned}
S &\rightarrow S\ S \\
S &\rightarrow a
\end{aligned}
$$

- Input: $aaaa$
- Derivation using sentential forms:
  $S \Rightarrow SS \Rightarrow aS \Rightarrow aSS \Rightarrow aaS \Rightarrow aaSS \Rightarrow aaaS \Rightarrow aaaa$
- But what about another derivation:
  $S \Rightarrow SS \Rightarrow SSS \Rightarrow SSSS \Rightarrow aSSS \Rightarrow \ldots \Rightarrow aaaa$
- Key problem with CFGs: **ambiguity**

Context-sensitive grammars: $L(G) = \{a^n b^n \mid n \geq 1\}$

$$
\begin{aligned}
S &\rightarrow S\ B\ C \\
S &\rightarrow a\ C \\
a\ B &\rightarrow a\ a \\
C\ B &\rightarrow B\ C \\
B\ a &\rightarrow a\ a \\
C &\rightarrow b
\end{aligned}
$$

Context-sensitive grammars: $L(G) = \{a^n b^n \mid n \geq 1\}$

$$
\begin{array}{r}
S_1 \\
S_2\ B_1\ C_1 \\
S_3\ B_2\ C_2\ B_1\ C_1 \\
a_3\ C_3\ B_2\ C_2\ B_1\ C_1 \\
a_3\ B_2\ C_3\ C_2\ B_1\ C_1 \\
a_3\ a_2\ C_3\ C_2\ B_1\ C_1 \\
a_3\ a_2\ C_3\ B_1\ C_2\ C_1 \\
a_3\ a_2\ B_1\ C_3\ C_2\ C_1 \\
a_3\ a_2\ a_1\ C_3\ C_2\ C_1 \\
a_3\ a_2\ a_1\ b_3\ b_2\ b_1
\end{array}
$$

Unrestricted grammars: $L(G) = \{a^{2i} \mid i \geq 1\}$

$$
\begin{aligned}
S &\rightarrow A\ C\ a\ B \\
C\ a &\rightarrow a\ a\ C \\
C\ B &\rightarrow D\ B \\
\mathbf{C\ B} &\rightarrow \mathbf{E} \\
a\ D &\rightarrow D\ a \\
A\ D &\rightarrow A\ C \\
a\ E &\rightarrow E\ a \\
\mathbf{A\ E} &\rightarrow \epsilon
\end{aligned}
$$

Unrestricted grammars: $L(G) = \{a^{2i} \mid i \geq 1\}$

$$\begin{array}{r} S \\ A\ C\ a\ B \\ A\ a\ a\ C\ B \\ A\ a\ a\ E \\ A\ a\ E\ a \\ A\ E\ a\ a \\ a\ a \end{array}$$

Unrestricted grammars: $L(G) = \{a^{2i} \mid i \geq 1\}$

- A and B serve as left and right end-markers for sentential forms (derivation of each string)
- C is a marker that moves through the string of a's between A and B, doubling their number using $C\ a \rightarrow a\ a\ C$
- When C hits right end-marker B, it becomes a D or E by $C\ B \rightarrow D\ B$ or $C\ B \rightarrow E$
- If a D is chosen, that D migrates left using $a\ D \rightarrow D\ a$ until left end-marker A is reached

Unrestricted grammars: $L(G) = \{a^{2i} \mid i \geq 1\}$

- At that point D becomes C using $A\ D \rightarrow A\ C$ and the process starts over
- Finally, E migrates left until it hits left end-marker A using $a\ E \rightarrow E\ a$
- Note that $L(G) = \{a^{2i} \mid i \geq 1\}$ can also be written as a context-sensitive grammar

Examples of Languages in the Chomsky Hierarchy

- **context-sensitive** grammars: 0^i, i is not a prime number and $i > 0$
- **indexed** grammars: $0^n 1^n 2^n \ldots m^n$, for any fixed m and $n \geq 0$
- **context-free** grammars: $0^n 1^n$ for $n \geq 0$
- **deterministic context-free** grammars: $S' \rightarrow S\ c$, $S \rightarrow S\ A \mid A$, $A \rightarrow a\ S\ b \mid ab$: the language of ”balanced parentheses”
- **regular** grammars: $(0|1)^*00(0|1)^*$

Language	*Automaton*	*Grammar*	*Recognition*	*Dependency*
Recursively Enumerable Languages	Turing Machine	Unrestricted Baa → A	Undecidable	Arbitrary
Context-Sensitive Languages	Linear-Bounded	Context-Sensitive At → aA	NP-Complete	Crossing
Context-Free Languages	Pushdown (stack)	Context-Free S → gSc	Polynomial	Nested
Regular Languages	Finite-State Machine	Regular A → cA	Linear	Strictly Local

Complexity of Parsing Algorithms

- Given grammar G and input x, provide algorithm for: Is $x \in L(G)$?
  - **unrestricted**: undecidable
  - **context-sensitive**: NSPACE(n) – linear non-deterministic space
  - **indexed** grammars: NP-Complete
  - **context-free**: $O(n^3)$
  - **deterministic context-free**: $O(n)$
  - **regular** grammars: $O(n)$

Verifying that $L = L(G)$

- Let's say we have a context-free grammar G and a description of a language L
- How can we say for sure that $L = L(G)$?
- By verifying the statement in two directions:
  $\Rightarrow$ All strings generated by G are in L
  $\Leftarrow$ All strings $w \in L$ can be generated by G

Verifying that $L = L(G)$

- Example: $T = \{a, b\}$. Consider language L to be "all strings with same number of as and bs"
- Consider G to be a CFG: $S \rightarrow \epsilon \mid a\ S\ b\ S \mid b\ S\ a\ S$
- To verify that $L = L(G)$, prove that
  $\Rightarrow$ All strings generated by G are in L
  $\Leftarrow$ All strings $w \in L$ can be generated by G

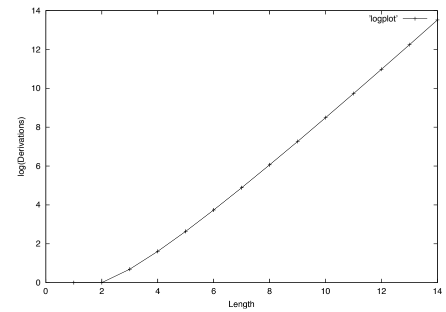
Proof ($\Rightarrow$): All strings generated by G are in L

- Proof by induction:
  - **Base case**: ϵ is in L (trivial)
  - **Inductive hypothesis**: Assume $u \in L$ and $v \in L$. Let w be generated by G with $|u| < |w|$ and $|v| < |w|$
    - Because w is generated by G then either $w \Rightarrow a\ u\ b\ v$ or $w \Rightarrow b\ u\ a\ v$, where u and v are generated by G
    - Since $|u| < |w|$ and $|v| < |w|$ and $u, v \in L$ then since we only added a single matching a, b pair, we can conclude that w is in L

Proof ($\Leftarrow$): All strings $w \in L$ can be generated by G

- Proof by induction (show that $S \Rightarrow^{+} w$):
  - **Base case**: $w = \epsilon$ (trivial: $S \rightarrow \epsilon$)
  - **Inductive hypothesis**: For a given $w \in L$, assume that for all $u, v \in L$ where $|u| < |w|$ and $|v| < |w|$ we have $S \Rightarrow^{+} u$ and $S \Rightarrow^{+} v$
    - ***Case 1* – w starts with a**: Find the first b from the right so that $w = a\ u\ b\ v$ and v has the same number of as and bs
      Because $w \in L$ it has to be true that $u, v \in L$ and by the inductive hypothesis $S \Rightarrow^{+} u$ and $S \Rightarrow^{+} v$
      Using rule $S \rightarrow a\ S\ b\ S$ and the above step we get $S \Rightarrow^{+} w$
    - ***Case 2* – w starts with b**: (analogous to Case 1)

CFG Ambiguity: Number of derivations grows exponentially



`L(G) = a+` using CFG rules { `S → S S`, `S → a` }

CFG Ambiguity

- Algebraic character of parse derivations
- Power Series for grammar for the (simplified) arithmetic expression CFG:
  `E → digit | E binop E`
- Write it down as an equation with coefficients equal to number of different analyses possible:

$$
\begin{aligned}
E &= \text{digit} + \text{digit}\ \mathtt{binop}\ \text{digit} \\
&+ 2(\text{digit}\ \mathtt{binop}\ \text{digit}\ \mathtt{binop}\ \text{digit}) \\
&+ 5(\text{digit}\ \mathtt{binop}\ \text{digit}\ \mathtt{binop}\ \text{digit}\ \mathtt{binop}\ \text{digit}) \\
&+ 14 \ldots
\end{aligned}
$$

CFG Ambiguity

- Coefficients in previous equation equal the number of parses for each string derived from E
- These ambiguity coefficients are Catalan numbers:

$$Cat(n) = \frac{1}{n+1}\binom{2n}{n}$$

- $\binom{a}{b}$ is the *binomial coefficient*

$$\binom{a}{b} = \frac{a!}{(b!(a-b)!)}$$

Catalan numbers

- Why Catalan numbers? Cat(n) is the number of ways to parenthesize an expression of length n with following conditions:

  1. there must be equal numbers of open and close parens
  2. they must be properly nested so that an open precedes a close
  3. the parentheses are used to encode groupings and spurious parenthesis groupings are not counted, e.g. `a(bc)` is counted but not `(a)(bc)`

Catalan numbers

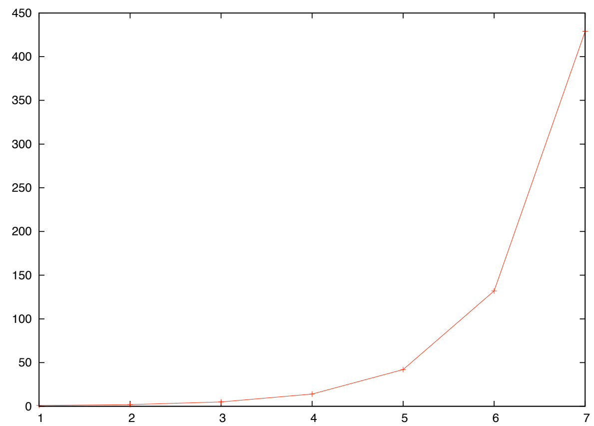
- For an expression of length n there are a total of $2n$ choose n parenthesis pairs, e.g. for 2 ops, $\binom{4}{2} = 6$: `a(bc)`, `)a(bc`, `a)bc(`, `(ab)c`, `)ab(c`, `ab)c(`
- But for each valid parenthesis pair, additional n pairs are created that have the right parenthesis to the left of its matching left parenthesis, from e.g. above: `)a(bc`, `a)bc(`, `ab)c(`
- So we divide $2n$ choose n by $n+1$:

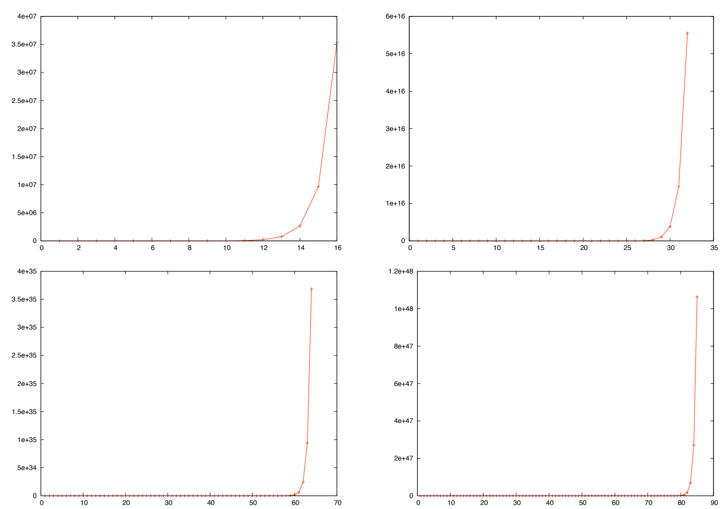
$$Cat(n) = \frac{\binom{2n}{n}}{n+1}$$

Catalan numbers

n	catalan(n)
1	1
2	2
3	5
4	14
5	42
6	132
7	429
8	1430
9	4862
10	16796

Catalan numbers





Summary

- Aspects of PL structure cannot be represented by FSAs
- Pumping lemma proofs for proving a language is not regular
- Chomsky hierarchy: from FSAs to Turing machines
- Verifying that a particular language is generated by a grammar G
- Context-free grammars (seems sufficient for PLs) but problems with ambiguity