On the structure of clone lattices, II

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To professor Ivo G. Rosenberg on his 65th birthday

Abstract. This survey paper is a continuation of [1] and aims to present some new directions and results obtained during the last years by the clone research group at Ural State University in Ekaterinburg.

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1 Introduction

Let $\mathcal{L}_k$ denote the lattice of clones on $E_k = \{0, 1, \ldots, k-1\}$. Many classification problems arising in different branches of universal algebra and theoretical computer science lead to the study of certain parts and properties of $\mathcal{L}_k$. A remarkable paper of I.G.Rosenberg [20] that was inspired by some problems in multiple-valued logic plays a great role in the study of algebraic properties of $\mathcal{L}_k$.

In [1], we presented our results obtained in the beginning of the nineties. Recall that those results concern clones of polynomials, monoidal intervals and lattices embeddable into $\mathcal{L}_k$. For further results in these directions we refer to [2, 3, 4, 5, 11, 12, 13, 14, 26].

In this paper as well as in [1] we study the structure of $\mathcal{L}_k$. Here we discuss some new directions our research has taken in recent years. We give a short motivation and formulate our results.

Clones with distributive subclone lattice. We show that any clone with a distributive subclone lattice is contained in a maximal such clone, describe possible unary parts of such clones and exhibit some examples of maximal clones with distributive subclone lattices.

Congruences of clone lattices. In this section we give a complete description of congruences of $\mathcal{L}_2$ and present some results concerning congruences of $\mathcal{L}_k$ for any finite k. In particular, we prove that $\mathcal{L}_k$ is directly indecomposable.

Ideals of iterative algebras. The notion of an iterative algebra (or closed set of functions) is a well-known generalization of the notion of a clone. We define ideals of iterative algebras and show that a description of iterative algebras without proper ideals can be reduced to a description of maximal surjective clones.

We consider these directions to be interesting and promising. We are especially pleased to mention that they were inspired by earlier papers of I.G.Rosenberg. We also are thankful to him for many fruitful discussions. Some of the presented results have already been published, while the others will be in forthcoming papers.

2 Clones with distributive subclone lattice

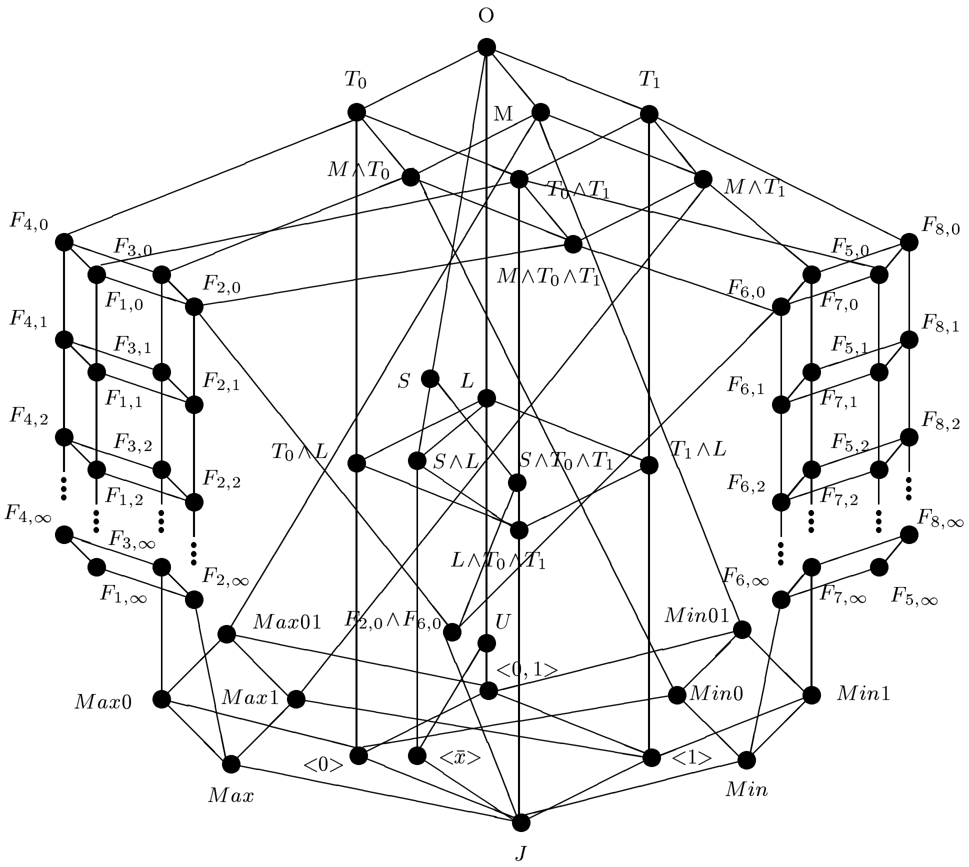
It is well known that the investigation of algebras whose subalgebra lattice satisfies some identity forces the development of techniques and results which make the structure of such algebras clearer. One of the earliest examples is the result of Ore [10] who proved that the subgroup lattice of a group is distributive if and only if the group is locally cyclic. For a detailed discussion of the advantage of the study of algebras with restrictions on subalgebra lattices we refer to [25, 27].

The first restriction on the subclone lattice that is worth studying is certainly the identity of distributivity. Clones with distributive subclone lattices (for short *distributive clones*) often appear in clone theory. For example, almost all known clones without finite basis [7, 28] are distributive (it follows from the main result of [9] that the clone given in [28] is distributive). Another example is given by the clone of all homogeneous functions [18]. In this section we give more examples of distributive clones and describe the unary part of a distributive clone. The general purpose of this investigation is to solve the following

Problem 1 *Describe all distributive clones.*

The results discussed in this section are contained in [6]. First of all, it can be easily checked that every distributive clone is contained in a maximal distributive clone (such a property holds for any algebra and any lattice identity). So, the solving of Problem 1 is reduced to finding maximal distributive clones.

In the Boolean case the maximal distributive clones can be identified by thouroughly looking at the diagram of $\mathcal{L}_2$ [19] (see the picture below). These are the clones $Max01, Min01, F_{4,1}, F_{8,1}, T_0 \wedge L, T_1 \wedge L, S \wedge L$.



We start the study of the general case by describing the monoid of unary functions from a distributive clone. We need some semigroup-theoretic notions.

A semigroup is said to be *left- (right-)zero* if it satisfies the identity $xy = x$ ($xy = y$). A semigroup S is an *ordinal sum* of subsemigroups $S_1, \ldots, S_n$ if $\{S_1, \ldots, S_n\}$ form a partition of S and $ab = ba = b$ for any $a \in S_i$, $b \in S_j$ where $i < j$. A subsemigroup I of a semigroup S is called an *ideal* if $ab \in I$ whenever one of a, b belongs to I. A semigroup S is an *ideal extension* of a semigroup S_1 by a semigroup S_2 with zero 0 if $S = S_1 \cup (S_2 \backslash \{0\})$ and S_1 is an ideal in S. Further, a semigroup S is said to be *nilpotent* (of degree n) if n is the smallest number such that $x_1 x_2 \ldots x_n = 0$ for any $x_1, x_2, \ldots, x_n \in S$. It is well known that some power of any element of a finite semigroup is an idempotent. For an idempotent e, the set of all elements a such that $a^n = e$ for some n is denoted by S_e. The set of all idempotents of a semigroup S is denoted by $E(S)$.

The monoid $\mathbf{C}^{(1)}$ of all unary functions of a clone $\mathbf{C}$ consists of permutations of a base set. We denote the range of a permutation g by Rg (g), while the kernel of g is denoted by ker (g). If a clone $\mathbf{C}$ is distributive, the monoid $\mathbf{C}^{(1)}$ has a distributive submonoid lattice. A description of such monoids can be derived from [24, 8].

Theorem 1 *Let $M = \mathbf{C}^{(1)}$ where $\mathbf{C}$ is a distributive clone.*

1. The set $E(M)$ is a submonoid of M and can be represented as an ordinal sum of left- and right-zero semigroups.

2. Any two idempotents from the same left-zero (right-zero) semigroup have the same kernel (range).

3. If $e, g \in E(M)$ belong to different components and $eg = ge = e$ then ker $(e) \subseteq$ ker (g) *and* Rg $(e) \supseteq$ Rg (f).

4. For any $e \in E(M)$, the subsemigroup M_e is an ideal extension of a cyclic group by a nilpotent semigroup of degree less than or equal to 5. Moreover, if the component of $E(M)$ containing e is not a singleton then the cyclic group is trivial.

We believe that the classification of distributive clones via the structure of the monoid $E(\mathbf{C}^{(1)})$ may lead to deep and fruitful results. Here we give some examples of maximal distributive clones with given monoid of idempotents. Let $k \geq 3$.

Example 1 *The clone $\mathbf{C_0}$ generated by all constant functions is a maximal distributive clone. The monoid $E(\mathbf{C_0})$ consists of two components. The* *first is formed by the identity transformation while the second is left-zero and consists of all constant operations.*

Example 2 *Let $B \subset E_k$, $|B| \geq 2$. By M_B we denote the set consisting of all functions of four types:*

(1) idempotents of the full transformation semigroup on E_k having the range B;

(2) functions $f(x_1, \ldots, x_n)$ with $\mathrm{Rg}\ (f) \subseteq B$ and such that for any $i \in \{1, \ldots, n\}$, $a_1, , \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in E_k$ and $a, b \in B$ the equality

$$f(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \tag{1}$$

holds;

(3) functions $f(x, x_1, \ldots, x_n)$ where $\mathrm{Rg}\ (f) \subseteq B$, $f(a, a_1, \ldots, a_n) = a$ whenever $a \in B$, and for any $i \in \{1, \ldots, n\}$, $c, a_1, , \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in E_k$, and $a, b \in B$ we have

$$f(c, a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) = f(c, a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \tag{2}$$

(4) projections and functions derived from them by permuting arguments.

The set M_B is a maximal distributive clone. The monoid $E(M_B^{(1)})$ consists of two components: one is formed by the identity transformation and the other one is a right-zero semigroup.

Example 3 *Let ψ be an equivalence relation having either at least two non-trivial classes, or a class containing at least three elements. By M_ψ we denote the set of all functions f satisfying the following conditions:*

– there is an i such that

$$f(a_1, \ldots, a_n) \equiv a_i \pmod{\psi}$$

for any $a_1, \ldots, a_n \in E_k$

– if $a_1 \equiv b_1 \pmod{\psi}, \ldots, a_n \equiv b_n \pmod{\psi}$ then

$$f(a_1, \ldots, a_n) = f(b_1, \ldots, b_n).$$

The set M_ψ is a maximal distributive clone and $E(M_\psi)$ consists of two components. The first is formed by the identity transformation and the other one is a left-zero semigroup.

3 Congruences of clone lattices

One of the possible ways to investigate $\mathcal{L}_k$ is to find some "smaller" lattices which reflect algebraic properties of the lattice. The best candidates would be homomorphic images of $\mathcal{L}_k$. For this reason congruences of clone lattices are of special interest. Moreover, certain partitions into convex subsets, elements with some special properties (e.g. distributive or neutral) and decomposability problems of lattices are connected with congruences. There are also reasons why congruences of clone lattices deserve special consideration.

The results discussed in this section are contained in [15, 16]. We start with the simplest case $k = 2$ in order to create a basis for further studies (for $k \geq 3$) and to find out what one could expect from them. By ∇ we denote the universal congruence on $\mathcal{L}_k$.

The lattice $\mathcal{L}_2$ is given by the diagram in the previous section. Let X and Y be partitions of the chain $\mathbb{N} \cup \{\infty\} = \{1, 2, 3, \ldots, \infty\}$ of type $\omega + 1$ (with natural order) into convex subsets. (Recall that a subset Q of an ordered set R is called convex if $a, b \in Q$ and $a \leq c \leq b$ imply $c \in Q$.) We define the equivalence $\theta_{X,Y}$ on $\mathcal{L}_2$ as follows. Let $C_1, C_2 \in \mathcal{L}_2$ be distinct. Then $C_1 \theta_{X,Y} C_2$ if and only if $\{C_1, C_2\} = \{F_{n,s}, F_{n,t}\}$ where $s, t \in \mathbb{N} \cup \{\infty\}$ and one of the following two conditions holds:

(a) $n \in \{1, 2, 3, 4\}$ and s, t belong to the same class of X;

(b) $n \in \{5, 6, 7, 8\}$ and s, t belong to the same class of Y.

Theorem 2 *Each relation of the form $\theta_{X,Y}$ is a congruence of $\mathcal{L}_2$. Moreover, any congruence of $\mathcal{L}_2$ distinct from ∇ is of this form.*

Denote by η the congruence $\theta_{X,Y}$ where both X and Y consist of a single class.

Corollary 1 *The relation η is the greatest non-trivial congruence of $\mathcal{L}_2$. The corresponding homomorphic image $\mathcal{L}_2/\eta$ is finite.*

Now we consider the case $k \geq 3$. The symbol θ will denote a congruence of $\mathcal{L}_k$ distinct from ∇.

Theorem 3 *Every clone which either contains all unary functions or is generated by unary functions forms a one-element class of θ.*

Corollary 2 *If $\mathcal{L}_k$ is not simple then it has a greatest non-trivial congruence.*

Let $\mathcal{L}_k^{\min}$ and $\mathcal{L}_k^{\max}$ denote the sets of all atoms and dual atoms of $\mathcal{L}_k$ respectively. These sets were classified in [21, 20].

Corollary 3 *Let $C_1 \theta C_2$. Then*
(i) for $C \in \mathcal{L}_k^{\min}$, $C \subseteq C_1$ if and only if $C \subseteq C_2$;
(ii) for $C \in \mathcal{L}_k^{\max}$, $C_1 \subseteq C$ if and only if $C_2 \subseteq C$;
(iii) C_1 and C_2 have the same unary part.

It can happen that $\mathcal{L}_k$ is simple. Now let us assume that this is not the case. Then, by Corollary 2, there exists a least non-trivial homomorphic image $\mathcal{L}_k^\star$. Hopefully, the lattice $\mathcal{L}_k^\star$ is less complex and has smaller cardinality than $\mathcal{L}_k$ (it may even be finite, as in the case $|A| = 2$ [15]). The structure of $\mathcal{L}_k^\star$ is still complex. For example, it follows from Corollary 3 that $\mathcal{L}_k^\star$ must have as many atoms and dual atoms as $\mathcal{L}_k$. Investigation of the lattice $\mathcal{L}_k^\star$ could give some new information about clone lattices and, possibly, provide us with some new technique for further studies.

The next statement exhibits some general properties of $\mathcal{L}_k$.

Theorem 4 *(i) $\mathcal{L}_k$ is directly indecomposable.*
(ii) $\mathcal{L}_k$ has neither distributive nor dually distributive elements except for the trivial ones.

No example of a non-trivial congruence of $\mathcal{L}_k$, $k \geq 3$, is known so far. Corollary 3 shows why such congruences (if they exist) are not easy to find. The conditions described there exclude many natural possible congruences. Thus the most intriguing open problem in this field is the following one.

Problem 2 *Does $\mathcal{L}_k$ have a non-trivial congruence for $k \geq 3$?*

The authors believe that this problem is of principal significance in the study of algebraic properties of clone lattices.

4 Ideals of iterative algebras

The notion of ideal has proved to be one of the most useful in semigroup theory. A clone can be considered as a special kind of semigroup. So, it is natural to consider ideals of clones. In this section we investigate this notion in application to iterative algebras (which are well-known generalization of clones).

Recall that an iterative algebra on E_k is a set of finitary functions on E_k closed under the binary operation $*$ of superposition and under four auxiliary operations (for the precise definition see [17]). Clones are exactly those iterative algebras which contain projections. Moreover, by adding projections to any iterative algebra we get a clone. Hence the lattice $\mathcal{L}_k$ is a principal ideal in the lattice $\mathcal{L}'_k$ of all iterative algebras on E_k.

We say that a subset $I \subseteq A$ of an iterative algebra A is an ideal of A if

(1) I is a subalgebra of A;

(2) I is an ideal of the semigroup $(A, *)$, i.e. $f * g \in I$ for any $f \in I$ and $g \in A$.

In semigroup theory, one distinguishes the notions of right ideal, left ideal and two-sided ideal. It is easy to check that any iterative algebra A has only trivial left ideals: $\emptyset$ and A. Thus, in the definition of ideal of an iterative algebra the term "ideal" means "right ideal". Note that a proper ideal of an iterative algebra cannot be a clone.

The notion of an ideal of an iterative algebra was first introduced in [22]. It was shown there by constructing examples that the set of ideals of an iterative algebra can be finite, countable and of continuum cardinality. Since the lattice $Idl(A)$ of all ideals of an iterative algebra A is distributive, there exists a poset Q such that $Idl(A)$ is isomorphic to the lattice of all down-sets of Q. For some important iterative algebras, such posets are determined in [22].

As in semigroup (and ring) theory, we say that an iterative algebra is simple if it has only trivial ideals. The study of simple iterative algebras was started in [23].

The following statement characterizes simple iterative algebras.

Theorem 5 *An iterative algebra A is simple if and only if all functions from A have the same range.*

Corollary 4 *(i) Any simple iterative algebra is contained in a maximal one.*

(ii) Simple iterative algebras on E_k form a down-set in the lattice $\mathcal{L}'_k$.

Corollary 5 *A clone is simple if and only if it consists of surjective functions.*

Now we show how to reduce a description of maximal simple iterative algebras to a description of maximal simple clones. Suppose A is a simple iterative algebra; then all functions from A have the same range U (in this case we write $Rg(A) = U$). It can be easily verified that the set $A|_U$ of restrictions $f|_U$ of all functions $f \in A$ forms a simple clone on U. If C is a simple clone on U then we denote by $[C]$ the set of all functions f on E_k such that $f|_U \in C$.

Theorem 6 *(i) Let $U \subseteq E_k$ and let C be a maximal simple clone on U. Then $[C]$ is a maximal simple algebra on E_k.*

(ii) Let A be a maximal simple iterative algebra with $Rg(A) = U$. Then $A|_U$ is a maximal simple clone on U, and we have $A = [A|_U]$.

It is easy to see that a clone on U consists of surjective functions if and only if its unary functions form a permutation group G on U. Recall that the stabilizer of G is the set

$$\begin{aligned} St(G) &= \{f \in O_k \mid f(g_1(x), \ldots, g_n(x)) \in G \\ &\qquad \text{for all } g_1, \ldots, g_n \in G\}. \end{aligned}$$

Suppose that $U_1 \cup \cdots \cup U_m = U$ is a partition of U, and G_i is a permutation group on U_i for $1 \le i \le m$. We denote by $G_1 \ldots G_m$ the permutation group on U consisting of all g such that $g|_{U_i} \in G_i$ for $1 \le i \le m$.

Theorem 7 *(i) Every maximal simple clone on U is of the form $St(G_1 \ldots G_m)$ for suitable partition $U_1 \cup \ldots \cup U_m$ of U and suitable transitive permutation groups G_i on U_i, $1 \le i \le m$.*

(ii) A clone of the form $St(G_1 \ldots G_m)$ is maximal simple if and only if $St(G_i)$ is a maximal simple clone on U_i, $1 \le i \le m$.

Therefore it remains to describe maximal simple clones of the form $St(G)$ where G is a transitive permutation group. One sufficient condition is given by the following statement.

Theorem 8 *If G is a regular permutation group then $St(G)$ is a maximal simple clone.*

With the help of the results above, all maximal simple iterative algebras on E_k are determined in [23] for $k \leq 4$.

We conclude this section with posing a natural problem which arises in this area and is interesting by itself.

Problem 3 *Describe all transitive permutation groups G such that $St(G)$ is a maximal clone of surjective functions.*

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