On the structure of clone lattices

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This paper is an account of investigations of clone lattices done during the last two years at the Ural University (Ekaterinburg).

Let $A = \{0, 1, \ldots, k-1\}$, and let $\mathcal{L}_k$ be the lattice of all clones of operations (or functions) on A. Our main purpose is the study of the algebraic structure of $\mathcal{L}_k$ or of certain its intervals. We examine the following three problems.

The stucture of the interval $[L_k^0, F_k]$. Let $k \geq 3$, and let

F_k be the clone of all polynomials over the ring Z_k of integers modulo k,

L_k the clone of all linear polynomials over Z_k,

F_k^0 the clone of all polynomials from F_k preserving the constant 0,

L_k^0 the clone of all linear forms over Z_k.

Our first task is to describe the interval $[L_k^0, F_k]$ in $\mathcal{L}_k$ through investigation of intervals $[L_k^0, F_k^0]$, $[L_k^0, L_k]$, $[F_k^0, F_k]$ and $[L_k, F_k]$. The corresponding results are due to E.Sukhanov, A.Krokhin and K.Safin.

Monoidal intervals. The second problem is the study of the structure of monoidal intervals in $\mathcal{L}_k$. Let M be a monoid of transformations of A. Recall that, for a clone $C \in \mathcal{L}_k$, the monoid (under composition) of all unary functions from C is called the *first layer* of C and is denoted by $C^{(1)}$. The set $\{C \in \mathcal{L}_k \mid C^{(1)} = M\}$ is an interval of $\mathcal{L}_k$. We denote it by $IntM$ as in [17] and call *monoidal*. The problem is to find the cardinality of $IntM$ for an arbitrary transformation monoid M. The corresponding results in this paper are due to A.Krokhin.

Embeddings into $\mathcal{L}_k$. The third problem is to find some wide classes of lattices embeddable into $\mathcal{L}_k$. In this direction, classes of finite and distributive lattices are of main interest. The corresponding results here are due to

A.Bulatov.

In the present paper we formulate and discuss our results. Their proofs will be published elsewhere. These results were announced (by E.V.Sukhanov) at the Fourth Conference on Discrete Mathematics in Potsdam, 1993 and we are grateful to participants of this conference for many helpful discussions. We are thankful especially to Professors K.Denecke, R. Pöschel and I.G.Rosenberg.

I. The structure of the interval $[L_k^0, F_k]$

First of all, it should be noted that this interval must be investigated for $k = p^n$ (p prime) only because if $k = p_1^{n_1} \cdots p_s^{n_s}$ (where p_i are distinct primes) then our interval is the direct product of similar intervals for $p_1^{n_1}, \ldots, p_s^{n_s}$. This follows from the well-known result (see e.g. [11], Th.3.3.3(ii)).

Proposition 1 *For any k, every clone from $[L_k^0, F_k]$ is defined completely by its intersections with F_k^0 and L_k.*

In other words, the interval $[L_k^0, F_k]$ (as a poset, but not as a lattice) is the subdirect product of the posets $[L_k^0, F_k^0]$ and $[L_k^0, L_k]$.

Recall that in the lattice of divisors $a \leq b$ iff a divides b.

Proposition 2 *Both intervals $[L_k^0, L_k]$ and $[F_k^0, F_k]$ are isomorphic to the lattice of divisors of k.*

So, for our purpose only the intervals $[L_k^0, F_k^0]$ and $[L_k, F_k]$ must be investigated. If we consider the case $k = p$, where p is a prime, then it is well known that the interval $[L_p, F_p]$ is prime (i.e. F_p covers L_p). But the situation is more complicated for the interval $[L_p^0, F_p^0]$.

Theorem 1 *For a prime p, the interval $[L_p^0, F_p^0]$ is isomorphic to the lattice $SubZ_{p-1}$ of all additive subgroups of Z_{p-1} (including the empty subgroup).*

So, for the simplest case $k = 3$, Propositions 1, 2 and Theorem 1 give the following picture:

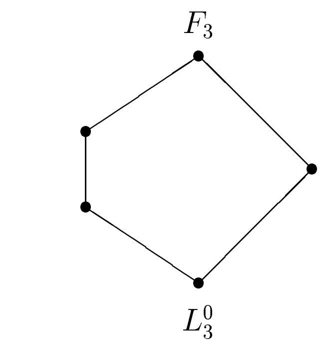

Figure 1.

In order to investigate the interval $[L_k, F_k]$ we have to consider the case $k = p^n$. The simplest situation here is $p = n = 2$, i.e. $k = 4$.

Theorem 2 *The interval $[L_4, F_4]$ is isomorphic to the lattice in Figure 2.*

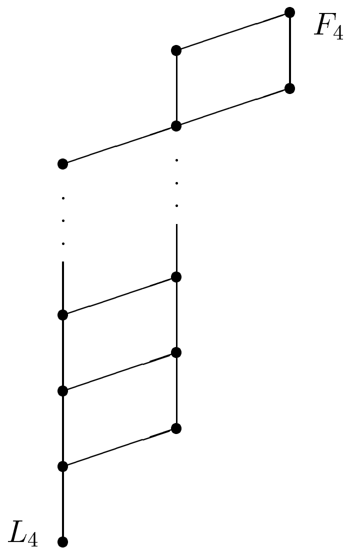

Figure 2.

The infinite chains on Figure 2 are $(\omega+1)$-ones.

In conclusion of this section we note that it seems to be of interest to study such intervals not only for Z_k, but for other rings. The interval from the clone of all linear functions over a finite field $GF(p^n)$ up to the clone of all polynomials over $GF(p^n)$ was described in [15].

II. Monoidal intervals

Let $O_k^{(n)}$ denote the set of all n-ary operations on A $(n = 1, 2, \ldots)$. Let $O_k = \bigcup_{n=1}^{\infty} O_k^{(n)}$. For $F \subseteq O_k$, let $\langle F \rangle$ denote the least clone containing F. Instead of $\langle \{f_1, \ldots, f_n\} \rangle$ we write simply $\langle f_1, \ldots, f_n \rangle$.

Evidently, for any clone C on A, its first layer $C^{(1)}$ is a transformation monoid on A with unit id, the identity transformation. Let M be an arbitrary transformation monoid on A. The *stabilizer* of M is the set

$$\begin{aligned} StM &= \{f \in O_k^{(n)} \mid n > 0 \text{ and } f(m_1(x), \ldots, m_n(x)) \in M \\ &\quad \text{for all } m_1, \ldots, m_n \in M\}. \end{aligned}$$

It's easy to see that StM is a clone. It is well known and easy to prove that $IntM = [\langle M \rangle; StM]$ (see [17], p.71). So, we have the partition of lattice $\mathcal{L}_k$ into intervals $IntM$ where M runs through all submonoids of $O_k^{(1)}$. The interval $IntM$ can be a singleton. This means that only one clone on A, namely $\langle M \rangle$, has M as its first layer. Such monoids M are called *collapsing*.

In the case $k = 2$ the lattice $\mathcal{L}_2$ is completely described in [12] and therefore we know all its monoidal intervals. Three of them are finite and the other three are countable. From now on in this section we shall always assume that $k \geq 3$. Then $|\mathcal{L}_k| = 2^{\aleph_0}$ (see [5]) and there is only finite number of transformation monoids on A. Hence there exists a monoid M such that $|IntM| = 2^{\aleph_0}$.

A.Szendrei poses in [17] the following problem: for which monoids M on A:

(a) $IntM$ is finite,

(b) $|IntM| = 2^{\aleph_0}$?

She states that the solution of this problem would contribute to a better understanding of the structure of $\mathcal{L}_k$. In [17] there is a series of examples of monoids M for which $IntM$ is a finite chain. Here are some of them: every transformation monoid containing all non-permutations, the monoid of all affine (linear) transformations of a finite vector space over a finite field, every nonregular transitive permutation group in which all nontrivial normal subgroups are transitive. I.A.Mal'cev proved in [7] that if $M = \{f(x) \mid |f(A)| \leq l\} \cup \{id\}$ $(2 \leq l \leq |A|)$ then $IntM$ is an $(l+1)$-element chain. Lemma 1.3.1 in [11] gives criteria for monoids and permutation groups to be collapsing. In [6] one can find another criterion for collapsing permutation groups. But, unfortunately, these criteria are hard to apply.

One result must be mentioned especially, because it has found an important application in universal algebra. P.P.Pálfy proved in [10] that if M consists of all constant operations and some permutations then $|IntM| \leq 2$ and, moreover, the equality holds iff M is the monoid of all unary polynomial operations of a vector space. In particular, the monoid consisting of all constants and the identity transformation is collapsing.

Here we consider all monoids M which consist of l constants and the identity operation. It follows from results of [8], [16] that $|Int\{id\}| = 2^{\aleph_0}$. As it is said above, in the case $l = |A|(= k)$, we have $|IntM| = 1$. The case $l = k - 1$ will be considered in Theorems 3 and 4, and the intervals $IntM$ turn out to be finite. If $1 \leq l \leq k - 2$ then M satisfies the conditions of Theorem 5 and we get $|IntM| = 2^{\aleph_0}$. This theorem and Corollary 2 from Theorem 6 yield a wide class of monoids M for which $|IntM| = 2^{\aleph_0}$ holds.

Now we formulate our theorems.

Let $A = \{0, 1, 2\}$. By $\min(x, y)$ and $\max(x, y)$ we denote the least and the greatest elements of $x, y \in A$ with respect to the order $0 < 2 < 1$.

Theorem 3 *$Int\{0, 1, id\}$ is the 4-element lattice shown in Figure 3.*

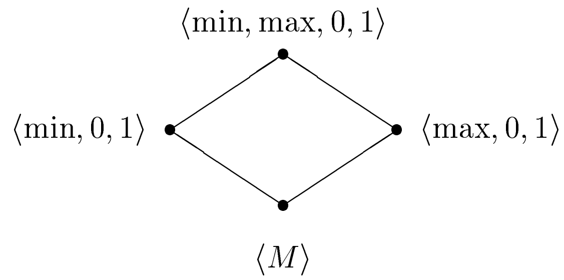

Figure 3.

Theorem 4 *If $k \geq 4$ and M consists of $k - 1$ constant operations and the identity operation then M is collapsing.*

Let $B \subseteq A$ satisfy $0 < |B| < k-1$. By B^A we denote the set of all transformations m of A such that $m(A) \subseteq B$.

Theorem 5 *If M is a monoid such that $M \subseteq B^A \cup \{id\}$ then $IntM$ has cardinality $2^{\aleph_0}$.*

Moreover, for every abstract finite nonsingleton monoid M', one can choose suitable finite set A and transformation monoid M on A isomorphic to M' such that M satisfies the conditions of Theorem 5.

Now let us consider a partition π of A into two nonvoid subsets A_0 and A_1, where $|A_0| \geq 2$. We denote by M_π the monoid of all operations m satisfying one of the following conditions:

(m1) $m(A_0) \subseteq A_0, m(A_1) \subseteq A_1$;

(m2) $m(A) \subseteq A_0$;

(m3) $m(A) \subseteq A_1$.

Every operation $^-$ satisfying the conditions $\overline{a} = a_0$ for all $a \in A_0$ and $\overline{a} = a_1$ for all $a \in A_1$, where a_0, a_1 are two different elements from A_0, will be called a *characteristic operation* of the partition π.

Theorem 6 *Let $M \subseteq M_\pi$ contain a characteristic operation of π. Then the lattice $IntM$ has an interval isomorphic to the countable direct power of the chain $(\omega + 1)$.*

Corollary 1 *Let M satisfy the conditions of Theorem 6. Then every direct product of at most $\aleph_0$ finite chains is isomorphic to a subinterval of $IntM$.*

If in Corollary 1 we take the countable direct power of the two-element chain then we get exactly the result from [4].

Corollary 2 *Let M satisfy the conditions of Theorem 6. Then $|IntM| = 2^{\aleph_0}$.*

Corollary 3 *([1]) Let $k \geq 3$. Then the number of clones in $\mathcal{L}_k$ containing all constants is $2^{\aleph_0}$.*

Clearly, the monoid M_π contains all constants and satisfies the conditions of Theorem 6. Corollary 3 follows from Corollary 2 if we set $M = M_\pi$.

In conclusion, many questions on monoidal intervals are still open. The most interesting among them is the following one: for $k \geq 3$, does there exist a countable monoidal interval?

III. Embeddings into $\mathcal{L}_k$

When we investigate such a lattice as complex as $\mathcal{L}_k$ we may pose various natural problems. One of them is the problem of finding families of its sublattices of some special kinds. This problem is solved for the lattice $\mathcal{L}_2$ because there exists complete description of $\mathcal{L}_2$ ([12]). For $k \geq 3$, there are only few advances. In particular, in [4] it was proved that a power set of a countable set can be embedded as an interval into $\mathcal{L}_3$ (and hence into $\mathcal{L}_k$ for $k \geq 3$ because $\mathcal{L}_3$ is isomorphic to some interval in $\mathcal{L}_k$). This fact implies that of any finite or countable distributive lattice is embeddable into $\mathcal{L}_3$. In [9] it was noted that, for any finite lattice L, there exists k (which may be rather large) such that L is a sublattice of $\mathcal{L}_k$. Further, as it was proved in [2], the subsemigroup lattice of the free semigroup of countable rank is embeddable into $\mathcal{L}_4$. Using this fact and some results of the papers [13], [14] we may conclude that there is a wide class of finite lattices which are embeddable into $\mathcal{L}_k$ for $k \geq 4$.

The starting-point is the question: for a given k, is every finite lattice embeddable into $\mathcal{L}_k$? A stronger assertion can be proved. It implies the solution for the problem of finding of all finite sublattices in $\mathcal{L}_k$ for $k \geq 4$.

Theorem 7 *Every direct product of at most countably many finite lattices is embeddable into $\mathcal{L}_4$.*

The theorem gives us examples of lattices of continuum cardinality embeddable into $\mathcal{L}_4$. If we consider only countable lattices of this kind then we get the well-known class of countable residually finite lattices.

Corollary 4 *Any countable residually finite lattice is embeddable into $\mathcal{L}_4$.*

Let $\mathcal{X}$ be a lattice variety generated by its finite members. It is well known (see e.g. [3], p.19) that the $\mathcal{X}$-free lattice of countable rank is residually finite. This fact gives an example of a sublattice of $\mathcal{L}_4$ for every variety $\mathcal{X}$ of the mentioned kind.

Corollary 5 *The $\mathcal{X}$-free lattice of countable rank is embeddable into $\mathcal{L}_4$.*

An embeddability of the absolutely free lattice of countable rank follows from results of [2], [13], [14].

Corollary 6 *Any finite lattice is embeddable into $\mathcal{L}_4$.*

Corollary 6 gives also the following important property of clone lattices.

Corollary 7 *The lattice $\mathcal{L}_4$ does not satisfy any nontrivial lattice quasi-identity.*

A similar assertion for nontrivial identities in $\mathcal{L}_3$ was proved in [2].

A natural question is whether the restriction $k \geq 4$ in the theorem is essential. So far we do not know any essential distinctions between the lattice properties of $\mathcal{L}_3$ and $\mathcal{L}_k$ for $k \geq 4$. The authors have no conjecture on this question.

Problem 1 *Can all finite lattices be embedded into $\mathcal{L}_3$?*

The following problem is closely connected with the preciding one and is of great interest.

Problem 2 *Does the lattice $\mathcal{L}_3$ satisfy any nontrivial quasi-identity?*

It would be very interesting to get some infomation about the embeddability of $\mathcal{L}_k$ into $\mathcal{L}_m$. Now, we know that $\mathcal{L}_k$ can be embedded as an interval into $\mathcal{L}_m$, if $m \geq k$ ([11], Th.3.3.3(i)). But, is the reverse embedding possible? More precisely, is $\mathcal{L}_m$ embeddable into $\mathcal{L}_k$ for some $3 \leq k < m$? We formulate a more concrete question.

Problem 3 *Is there a $k \geq 3$ such that every $\mathcal{L}_m$ can be embedded into $\mathcal{L}_k$? If yes, can we choose $k = 3$ or $k = 4$?*

By Corollary 6 any finite lattice is isomorphic to some sublattice of $\mathcal{L}_4$. However it is doubtful whether this is true for countable lattices. Lattices embedded into $\mathcal{L}_4$ by Theorem 7 have many congruences. So, we should find an infinite lattice, which is not embeddable into $\mathcal{L}_4$ and has only a few congruences, for example, a simple lattice. The best known lattices of this kind are the partition lattice of on a countable set and the lattice M_ω, i.e. the lattice of height 2 with a countable set of incomparable elements.

Problem 4 *Is the lattice M_ω or the prtition lattice of a countable set embeddable into $\mathcal{L}_k$ for some k (in particular, for $k = 3, 4$)?*

Theorem 6, Corollary 1 and the above-mentioned result of [4] give us examples of lattices which can be embedded into $\mathcal{L}_k$ as an interval. So, we can reformulate the above problems for embedding of such restricted kind.

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