On one semigroup property of clones

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Let A be a finite set of cardinality k. The set of all finitary functions on A is denoted by P_A. Usually the only essential thing is the cardinality of A so we often write P_k instead of P_A. We consider P_A with operations of superpositions of functions, permutation and identifying of variables, and adding of fictitious arguments. Recall that a variable x_i of a function $f(x_1,\dots,x_n)$ is said to be essential if there exists $a, b, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in A$ such that $f(a_1,\dots,a_{i-1},a,a_{i+1},\dots,a_n) \neq f(a_1,\dots,a_{i-1},b,a_{i+1},\dots,a_n)$. A variable is called fictitious if it is not essential.

All operations mentioned above can be obtained from five operations only. Here is the list of these operations. Let $f(x_1,\dots,x_n), g(x_1,\dots,x_m)$ be functions. Then we put

1) $(f \circ g)(x_1,\dots,x_{m+n-1}) = f(g(x_1,\dots,x_m), x_{m+1},\dots,x_{m+n-1})$ — substitution of g for the first variable of f;

if $n \geq 2$, then

2) $(\Delta f)(x_1,\dots,x_{n-1}) = f(x_1,x_1,x_2,\dots,x_{n-1})$ — identification of variables;

3) $(\nabla f)(x_1,\dots,x_{n+1}) = f(x_2,\dots,x_{n+1})$ — adding of a fictitious variables;

4) $(\pi f)(x_1,\dots,x_n) = f(x_2,x_3,\dots,x_n,x_1)$ and

5) $(\xi f)(x_1,\dots,x_n) = f(x_2,x_1,x_3,\dots,x_n)$ — permutations generating the symmetric group on the set of variables of f;

finally, if $n = 1$, then $\pi f = \xi f = \Delta f = f$.

Any algebra $\langle C; \circ, \Delta, \nabla, \pi, \xi\rangle$, where $C \subseteq P_A$ is called an iterative algebra. This notion has been introduced by A.I.Mal'tsev [?]. Further, a clone is an iterative algebra containing all projections, that is functions of the form $e_n^i(x_1,\dots,x_n) = x_i$ for some n and i.

The usual approaches to investigation on clones are the following. One consider clones of functions of some specific kind or clones containing some special functions, for example, Mal'tsev function. In these approaches properties of functions are primary.

However, the operation of superposition (operation $\circ$) is associative, so we can treat any clone as a semigroup with some additional unary operations. From this point of view it is very naturally to consider semigroup properties of clones or, in other words, properties of the operation $\circ$. The author believes that the approach to consider clones as semigroups is very useful.

Now we deal with one of such properties, namely, the property to be a semigroup with zero multiplication. Unfortunately, since functions can have fictitious variables, a number of semigroup properties of clones are either trivial or not very useful. For instance, if the first variable of a function is a fictitious one, then the function behaves like a left zero under the operation $\circ$. This fact implies, in particular, that there is no clone being a semigroup with zero multiplication. So, we need a slight correction of this notion for clones. The following notion is a very natural analogue of the notion of zero multiplication semigroup. A clone C is said to be a clone of zero square if for all $f, g \in C$ such that f, g differ from projections and the first variable of f isn't fictitious, the function $f \circ g$ is a constant function. Thus constant functions play role of zeroes and C may contain several (left) zeroes.

The result formulated below is concerned with maximal clones of zero square. Let D and B be sets satisfying the conditions

$$\varnothing \neq D \subseteq B \subset A, \quad |B| > 1,$$

π be a partition of B and every class of π contains an element of D. Denote by $Z_{D,\pi,B}$ the clone consisting of the following functions

I. functions $f(x_1, \ldots, x_n)$ satisfying the conditions:

   (1) image of f is contained in a class of π;

   (2) there exists $c \in D$ such that for all $a_1, \ldots, a_n \in A$ if for some $i \in \{1, \ldots, n\}$ the variable x_i is essential and $a_i \in B$, then $f(a_1, \ldots, a_n) = c$;

II. unary functions g satisfying condition (1) and such that $g(B) \subseteq D$ and g is a constant on each class of π;

III. all projections.

As we see D is the set of zeroes of clone $Z_{D,\pi,B}$.

First example of a clone of such kind is the well-known clone that have been constructed by Yanov and Muchnik [?]. Let $A = \{0, 1, 2\}$. This clone consists of functions of the form

$$f_n(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } x_1 = \ldots = x_n = 2 \\ 0 & \text{otherwise} \end{cases}$$

(such functions is sometimes said to be functions with a punctured point) and functions that are derived from them by adding of fictitious variables. It coincides with the clone $Z_{D,\pi,B}$ where $D = \{0\}$, $B = \{0, 1\}$ and π has the only class $\{0, 1\}$.

Theorem 1 *Every clone of zero square is a subclone of a maximal one. The clones $Z_{B,\pi,B}$ where B is a nonsingleton subset of A are exactly the maximal clones of zero square having B as the set of zeroes.*

Proof. It is easy to see, that, for any $B \subseteq A$ and for any partition π of B, the clone $Z_{B,\pi,B}$ is a clone of zero square. Indeed, take $f(x_1, \ldots, x_n)$, $g(x_1, \ldots, x_m) \in Z_{B,\pi,B}$. Suppose that f is of type I, x_1 is an essential variable and let c be the constant from condition (2). Then, for any $a_1, \ldots, a_{m+n-1} \in A$, we have $f(g(a_1, \ldots, a_m), a_{m+1}, \ldots, a_{m+n-1}) = c$ since $g(a_1, \ldots, a_m) \in B$, i.e. $f \circ g$ is a constant. Further, suppose that f is a unary function of the type II. In this case Im g (Im g denotes the image of g) is contained in a class of π and $f\|_{\mathrm{Im}\ g}$ is a constant function. Hence, $f \circ g$ is a constant function.

Now let C be a clone of zero square. Recall that $C^{(1)}$ denotes the set of unary functions from C and $\circ_i$ denotes the operation of substitution for i-th variable. Put $B = \bigcup_{f \in C} \mathrm{Im}\ f$ and $\pi = \bigcap_{g \in C^{(1)}} \ker g$. We prove that $C \subseteq Z_{B,\pi,B}$. Take $f(x_1, \ldots, x_n) \in C$ and $b \in B$. There are $g(x_1, \ldots, x_m) \in C$ and $a_1, \ldots, a_m \in A$ such that $g(a_1, \ldots, a_m) = b$. If x_i is an essential variable of f, then the function $f \circ_i g$ is a constant, consequently, variables of the function $f(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, g(a_1, \ldots, a_m), x_{i+1}, \ldots, x_n)$ are fictitious. So, $f(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n) = c$, where $c \in B$.

Suppose that there exists another essential variable x_j of f (without loss of generality we can assume that $i < j$). Take $b_1, b_2 \in B$ and put $c_1 = f(x_1, \ldots, x_{i-1}, b_1, x_{i+1}, \ldots, x_n)$, $c_2 = f(x_1, \ldots, x_{j-1}, b_2, x_{j+1}, \ldots, x_n)$. The variable x_j is a fictitious one of $f(x_1, \ldots, x_{i-1}, b_1, x_{i+1}, \ldots, x_n)$ and x_i

is a fictitious variable of $f(x_1,\ldots,x_{j-1},b_2,x_{j+1},\ldots,x_n)$. Hence

$$\begin{aligned} c_1 &= f(x_1,\ldots,x_{i-1},b_1,x_{i+1},\ldots,x_n) = \\ & f(x_1,\ldots,x_{i-1},b_1,x_{i+1},\ldots,x_{j-1},b_2,x_{j+1},\ldots,x_n) \\ & f(x_1,\ldots,x_{j-1},b_2,x_{j+1},\ldots,x_n) = c_2. \end{aligned}$$

So, $c_1 = c_2$. Moreover, for every $b_3 \in B$ we have

$$\begin{aligned} & f(x_1,\ldots,x_{i-1},b_1,x_{i+1},\ldots,x_n) = \\ & \quad = f(x_1,\ldots,x_{j-1},b_2,x_{j+1},\ldots,x_n) = \\ & \quad = f(x_1,\ldots,x_{i-1},b_3,x_{i+1},\ldots,x_n). \end{aligned}$$

Thus if f has at least two essential variables, then there exists $c \in B$ such that for all $a_1,\ldots,a_n \in A$ if there is an element $a_i \in B$ and x_i is an essential variables, then $f(a_1,\ldots,a_n) = c$. It proves property (2).

To end the proof take $f \in C$, $g \in C^{(1)}$ and $a, b \in \operatorname{Im} f$, that is $a = f(a_1,\ldots,a_n)$, $b = f(b_1,\ldots,b_n)$ for some $a_1,\ldots,a_n,b_1,\ldots,b_n \in A$. We have $g \circ f = \mathrm{const}$, consequently, $(g \circ f)(a_1,\ldots,a_n) = (g \circ f)(b_1,\ldots,b_n)$. Hence, $g(a) = g(b)$ and $(a,b) \in \ker g$. It means $(a,b) \in \pi$, since g is an arbitrary unary function from C. So, $\operatorname{Im} f$ is contained in a class of π and property (1) is checked. Finally, by consruction of π, any unary function $g \in C$ is constant on every class of π. The theorem is proved. □

So, we see that there are only finite number of maximal clones with zero square. We make some remarks about clones of the form $Z_{D,\pi,B}$. Since a clone is a universal algebra we may speak about a generating set of a clone in the usual sense. For instance, the mentioned above clone, which have been constructed by Yanov and Muchnik, has no independent generating set (and it was the first such example). The same property holds for a clone $Z_{D,\pi,B}$ for any D, B and π.

Another remark is connected with clone lattices. A clone $Z_{D,\pi,B}$ has a distributive lattice of subclones if and only if π has the only class. It is of great interest that we can extend each clone $Z_{B,\{B\},B}$ to the maximal clone with distributive lattice of subclones.

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References

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