Polynomial reducts of modules.II. Algebras of primitive and nilpotent functions

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Abstract. The paper is a continuation of the study of iterative algebras of linear functions on a finite module M which was started in [1]. Let $f = \sum \alpha_i x_i + b$ be a linear function with coefficients from a finite ring R. The function f is called primitive if one of irs coefficient is invertible and others are nilpotent. We say that the function f is nilpotent if all its coefficients are nilpotent. Iterative algebras which consist of primitive functions are finite in number for any R and any M Also, we consider special sets of iterative algebras of primitive and nilpotent functions. Each of these sets forms an interval of the lattice of iterative algebras and consists of algebras which contain the same primitive functions. Finite intervals of this form are described.

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Let us recall some definitions and notation. A set $\mathbf{C}$ of functions on a finite set A is called an iterative algebra if $\mathbf{C}$ is closed under operations $\circ$, Δ, ∇, τ, ξ where $\circ$ is substitution of a function instead of the first variable of another one, Δ is identification of first and second variables, ∇ is adding of fictitious variables and τ, ξ are certain permutations of variables such that τ, ξ generate the group of all permutations (for details see e.g. [2]).

Let R be a finite associative commutative ring with an identity element 1 and let M be a finite faithful unitary R-module. A linear operation on M is a polynomial operation of the module M. The set of all linear functions is denoted by $\mathbf{L}_M$. The set $\mathbf{L}_M$ is an iterative algebra and Sub $\mathbf{L}_M$ denotes the lattice of all its sualgebras. In [1], the partition of Sub $\mathbf{L}_M$ into four types was introduced. The present paper is devoted to study of one of these parts.

A function $f = \alpha_1 x_1 + \ldots + \alpha_n x_n + b \in \mathbf{L}_M$ is called nilpotent if $\alpha_1, \ldots, \alpha_n$ are nilpotent. The function is called primitive if there is a unique invertible element among $\alpha_1, \ldots, \alpha_n$ and the others are nilpotent elements. A primitive (resp. nilpotent) algebra is an algebra which consists of primitive (resp. nilpotent) functions. The algebra of all primitive functions and the algebra of all nilpotent functions are denoted by $\mathbf{LP}_M$ and $\mathbf{N}_M$ respectively. The interval $[\varnothing, \mathbf{LP}_M \vee \mathbf{N}_M]$ is the mentioned class of the partition of Sub $\mathbf{L}_M$. For details the reader is referred to [1].

We denote the arity of a function f by ar$f$ and the set $\{1, \ldots, n\}$ by $\underline{n}$. For functions f, g where ar$f = n$, ar$g = m$, $i, j \in \underline{n}$, $i < j$ and for a permutation π of the set $\underline{n}$, we put

$$\begin{aligned}
f \circ_j g &= f(x_1, \ldots, x_{j-1}, g(x_j, \ldots, x_{j+m-1}), x_{j+m}, \ldots, x_{n+m-1}) \\
&\qquad \text{(substitution instead of the } j\text{-th variable)} \\
(\pi f)(x_1, \ldots, x_n) &= f(x_{\pi(1)}, \ldots, x_{\pi(n)}) \\
&\qquad \text{(permutation of variables)} \\
(\Delta_{i,j} f)(x_1, \ldots, x_{n-1}) &= f(x_1, \ldots, x_i, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_{n-1}) \\
&\qquad \text{(identification of } i\text{-th and } j\text{-th variables)}
\end{aligned}$$

We denote the algebra of all projections (i.e. functions of the form $e_i^n(x_1, \ldots, x_n) = x_i$) by $\mathbf{J}_M$. The set of all nilpotent elements of R is denoted by Nil$\,(R)$.

§1. The lattice of primitive algebras

To describe the interval $[\mathbf{J}_M, \mathbf{LP}_M]$ we find a one-to-one correspondence between $[\mathbf{J}_M, \mathbf{LP}_M]$ and certain set of pairs (S, G) where S is a subring of Nil$\,(R)$ and G is a subalgebra of a suitable finite algebra which is close to a module. Using this correspondence we prove that the lattice of primitive algebras is finite. Note that, on the other hand, the lattice of nilpotent algebras and the whole lattice Sub $\mathbf{L}_M$ are infinite, provided R has a non-zero nilpotent element.

Given $f = \sum_{i=1}^n \alpha_i x_i + b \in \mathbf{L}_M$, the pair $\mathrm{Tr}(f) = (\sigma(f), \beta(f))$ where $\sigma(f) = \sum_{i=1}^n \alpha_i - 1$, $\beta(f) = b$ is called the *trace* of f. For an iterative algebra $\mathbf{C} \subseteq \mathbf{L}_M$, $\mathrm{Tr}(\mathbf{C})$ denotes the set $\{\mathrm{Tr}(f) \mid f \in \mathbf{C}\}$. The set of nilpotent coefficients of a primitive function g is denoted by Nil$\,(f)$. For a primitive algebra $\mathbf{Q}$, the subring of Nil$\,(R)$ generated by the set $\bigcup_{f \in \mathbf{Q}}$ Nil$\,(f)$, is denoted by Nil $(\mathbf{Q})$. We prove that each primitive algebra $\mathbf{Q}$ is uniquely determined by the pair $(\mathrm{Tr}(\mathbf{Q}), \mathrm{Nil}(\mathbf{Q}))$.

Lemma 1 *If $\varrho \in \mathrm{Nil}(\mathbf{Q})$, then the function $s_\varrho(x, y, z) = x - \varrho y + \varrho z$ belongs to $\mathbf{Q}$.*

Proof.

Let us start with the case when ϱ is a coefficient of a certain function of $\mathbf{Q}$. Then there are $\alpha \in R$ and $b \in M$ such that the function $f = \alpha x + \varrho y + b$ belongs to $\mathbf{Q}$. Since α is invertible we have $\alpha^n = 1$ for some n. Identifying variables $y_2, \ldots, y_n$ of the function

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n-1 \text{ times}} = \alpha^n x + \varrho y_1 + \alpha\varrho y_2 + \ldots + \alpha^{n-1}\varrho y_n + (1 + \alpha + \ldots + \alpha^{n-1})b$$

we obtain

$$g = x + \varrho y + \gamma z + b'.$$

Let t be an integer such that $t\sigma(g) = 0$ and $tb' = 0$. Then

$$g^t = x + \varrho \sum_{i=1}^{t} y_i + \gamma \sum_{i=1}^{t} z_i + tb' \in \mathbf{Q}$$

and

$$h = x + \varrho y + \vartheta z \in \mathbf{Q}$$

where h arises from g^t by identification of $y_2, \ldots, y_t, z_1, \ldots, z_t$. Since $\sigma(h) = \sigma(g^t) = t\sigma(g) = 0$, we have $\vartheta = -\varrho$. Thus the function $s_\varrho = x - \varrho y + \varrho z$ belongs to $\mathbf{Q}$.

Finally,

$$s_{\varrho_1+\varrho_2}(x, y, z) = s_{\varrho_1}(s_{\varrho_2}(x, y, z), y, z)$$

and

$$s_{\varrho_1\varrho_2}(x, y, z) = s_{\varrho_1}(x, y, s_{\varrho_2}(y, y, z)) =$$
$$x - \varrho_1 y + \varrho_2 y - \varrho_1\varrho_2 y + \varrho_1\varrho_2 z = .x - \varrho_1\varrho_2 y + \varrho_1\varrho_2 z$$

for any $\varrho_1, \varrho_2 \in \mathrm{Nil}(\mathbf{Q})$. Thus, for any $\varrho \in \mathrm{Nil}(\mathbf{Q})$, we have $s_\varrho \in \mathbf{Q}$. Lemma 1 is proved. $\square$

Lemma 2 *Let* $\mathbf{Q}$ *be a primitive algebra. A primitive function* g *belongs to* $\mathbf{Q}$ *if and only if both* $\mathrm{Tr}(g) \in \mathrm{Tr}(\mathbf{Q})$ *and* $\mathrm{Nil}\,(g) \subseteq \mathrm{Nil}\,(\mathbf{Q})$.

Proof.

If $g \in \mathbf{Q}$, then $\mathrm{Tr}(g) \in \mathrm{Tr}(\mathbf{Q})$, $\mathrm{Nil}\,(g) \subseteq \mathrm{Nil}\,(\mathbf{Q})$ by definition. So suppose $\mathrm{Tr}(g) \in \mathrm{Tr}(\mathbf{Q})$ and $\mathrm{Nil}\,(g) \subseteq \mathrm{Nil}\,(\mathbf{Q})$. Then there exists $f \in \mathbf{Q}$ such that $\mathrm{Tr}(g) = \mathrm{Tr}(f)$. Let $f = \sum_{i=1}^{m} \alpha_i x_i + a$ and $g = \sum_{i=1}^{n} \gamma_i x_i + b$. Then $a = b$ because $\beta(f) = \beta(g)$. Identifying all variables of f, we obtain $f' = (\sigma(f) + 1)x + a$. Further, the function f' is primitive, so $\sigma(f) + 1 \neq 0$. Suppose γ_j is an invertible element. By Lemma 1 the function s_{γ_i} belongs to $\mathbf{Q}$ for any $i \in \underline{m}\backslash\{j\}$. Hence the function

$$h = (\sigma(f) + 1)x - \sum_{i \in \underline{m}\backslash\{j\}} \gamma_i y_i + \sum_{i \in \underline{m}\backslash\{j\}} \gamma_i z_i + b$$

belongs to $\mathbf{Q}$. Identifying the variables x and y_i of h where $i \in \underline{m}\backslash\{j\}$, we obtain

$$\overline{h} = (\sigma(f) - \sum_{i \in \underline{m}\backslash\{j\}} \gamma_i + 1)x_j + \sum_{i \in \underline{m}\backslash\{j\}} \gamma_i y_i + b \in \mathbf{Q}.$$

Since $\gamma_j + \sum_{i \in \underline{m}\backslash\{j\}} \gamma_i - 1 = \sigma(g) = \sigma(f)$, we have $\gamma_j = \sigma(f) - \sum_{i \in \underline{m}\backslash\{j\}} \gamma_i + 1$. Therefore, $\overline{h} = \gamma_j x_j + \sum_{i \in \underline{m}\backslash\{j\}} \gamma_i x_i + b = g$. This completes the proof of Lemma 2. $\Box$

Let $\mathbf{Q}$ be a primitive algebra and let $\diamond$ and $*_\varrho$, for $\varrho \in \mathrm{Nil}\,(R)$, be the operations on $\mathrm{Tr}(\mathbf{Q})$ defined by the rules:

$$\begin{aligned} \diamond &: \ (\alpha_1, b_1) \diamond (\alpha_2, b_2) = (\alpha_1 + \alpha_1\alpha_2 + \alpha_2, b_1 + \alpha_1 b_2 + b_2); \\ *_\varrho &: (\alpha_1, b_1) *_\varrho (\alpha_2, b_2) = (\alpha_1 + \varrho\alpha_2, b_1 + \varrho b_2) \end{aligned}$$

where $(\alpha_1, b_1), (\alpha_2, b_2) \in \mathrm{Tr}(\mathbf{Q})$.

Lemma 3 *The set* $\mathrm{Tr}(\mathbf{Q})$ *is closed under* $\diamond$ *and* $*_\varrho$ *for all* $\varrho \in \mathrm{Nil}\,(\mathbf{Q})$.

Proof.

Let $(\alpha_1, b_1) = \mathrm{Tr}(f_1)$ and $(\alpha_2, b_2) = \mathrm{Tr}(f_2)$ where $f_1, f_2 \in \mathbf{Q}$. The unary functions $f_1' = (\sigma(f_1) + 1)x + b_1$, $f_2' = (\sigma(f_2) + 1)x + b_2$ arise from f_1 and f_2

by identifying of variables and $\mathrm{Tr}(f_1') = \mathrm{Tr}(f_1)$, $\mathrm{Tr}(f_2') = \mathrm{Tr}(f_2)$. For f_1', f_2' and arbitrary $\varrho \in \mathrm{Nil}\,(\mathbf{Q})$, we have

$$\begin{aligned}&\mathrm{Tr}(f_1' \circ f_2') = \mathrm{Tr}(f_1') \diamond \mathrm{Tr}(f_2'),\\ &\mathrm{Tr}(s_\varrho(f_1(x), x, f_2(x)) = \mathrm{Tr}(f_1') *_\varrho \mathrm{Tr}(f_2').\end{aligned}$$

So $\mathrm{Tr}(f_1) \diamond \mathrm{Tr}(f_2) \in \mathrm{Tr}(\mathbf{Q})$ and by Lemma 3 $\mathrm{Tr}(f_1) *_\varrho \mathrm{Tr}(f_2) \in \mathrm{Tr}(\mathbf{Q})$. □

Now we describe sets of the form $\mathrm{Tr}(\mathbf{Q})$.

Let S be a subring of $\mathrm{Nil}\,(R)$. The set of all invertible elements α such that $\alpha S \subseteq S$ is denoted by $U_R(S)$. In other words $U_R(S)$ is the set of all invertible elements from the idealizer of S (that is the set of all $\alpha \in R$ such that $\alpha S \subseteq S$). It is easy to see that $U_R(S)$ is a subgroup of the group of invertible elements and $S \subseteq U_R(S) - 1 = \{\alpha - 1 \mid \alpha \in U_R(S)\}$. We denote by G_S an algebra with the universe $(U_R(S) - 1) \times M$ endowed with the operations $\diamond$ and $\{*_\varrho \mid \varrho \in S\}$. It is not hard to show that these operations preserve the set $(U_R(S) - 1) \times M$. The operation $\diamond$ is associative. Moreover, the restrictions of $\diamond$ to the first factor and of the adjoint multiplication of R to $U_R(S) - 1$ coincide. Since $\alpha_1 + \alpha_1\alpha_2 + \alpha_2 = (\alpha_1 + 1)(\alpha_2 + 1) - 1$, the set $U_R(S) - 1$ is an Abelian group with respect to $\diamond$.

Lemma 4 *If S is a subring of* $\mathrm{Nil}\,(R)$ *and $G \neq \varnothing$ is a subalgebra of G_S, then* $(0, 0) \in G$.

Proof.

First, the operation $\diamond$ can be written in the form $(\alpha_1, b_1) \diamond (\alpha_2, b_2) = ((\alpha_1 + 1)(\alpha_2 + 1) - 1, b_1 + (\alpha_1 + 1)b_2)$. Since $G \neq \varnothing$, there is $(\alpha, b) \in G$. Let $t = t'|R|$ where t' is multiplicative order of $\alpha + 1$. Then

$$\begin{aligned}(\alpha, b)^t &= \underbrace{(\alpha, b) \diamond \cdots \diamond (\alpha, b)}_{t \text{ times}} = ((\alpha + 1)^t - 1, b(1 + (\alpha + 1) + \ldots + \\ &\qquad +(\alpha + 1)^{t-1}) = \\ &= (0, b(|R| \cdot 1 + |R|(\alpha + 1) + \ldots + |R|(\alpha + 1)^{t'-1})) = (0, 0),\end{aligned}$$

as desired. □

We need some notation. Let $\mathcal{P}_M$ denote the set of all pairs (S, G) where S is a subring of $\mathrm{Nil}\,(R)$ and G is a subalgebra of G_S. Let S_1, S_2 be subrings of $\mathrm{Nil}\,(R)$ and let G_1, G_2 be subalgebras of G_{S_1} and G_{S_2}, respectively. We put $(S_1, G_1) \leq (S_2, G_2)$ if and only if $S_1 \subseteq S_2$ and $G_1 \subseteq G_2$. The set $\mathcal{P}_M$ is a partial ordered set with respect to $\leq$. For a subalgebra G of G_S, the set $\{\alpha \in U_R(S) \mid \text{there exists } b \in M \text{ such that } (\alpha - 1, b) \in G\}$ is denoted by $\mathrm{Pr}(G)$. The set $\mathrm{Pr}(G)$ is close to the projection to the first component of G. Finally, let $\langle G \rangle_{G_S}$ denote the subalgebra of G_S generated by a set $G \subseteq (U_R(S) - 1) \times M$ and let $\langle T \rangle_R$ stand for the subring of R generated by $T \subseteq R$.

Proposition 1 *The partial order of $\mathcal{P}_M$ is a lattice one and for the lattice operations the following formulas are valid*

(1) $(S_1, G_1) \wedge (S_2, G_2) = (S_1 \cap S_2, G_1 \cap G_2)$ where $G_1 \cap G_2$ is considered as a subalgebra of $G_{S_1 \cap S_2}$;

(2) $(S_1, G_1) \vee (S_2, G_2) = (S', \langle G_1 \cup G_2 \rangle_{G_{S'}})$ where $S' = \langle S_1 \cdot \mathrm{Pr}(G_2) \cup S_2 \cdot \mathrm{Pr}(G_1) \rangle_R$.

Proof.

To prove (1) take $(\alpha - 1, b) \in G_1 \cap G_2$. Since $\alpha \in U_R(S_i)$ for $i = 1, 2$, we have $\alpha(S_1 \cap S_2) \subseteq \alpha S_i \subseteq S_i$, i.e. $\alpha(S_1 \cap S_2) \subseteq S_1 \cap S_2$. Hence, $\alpha \in U_R(S_1 \cap S_2)$ and $G_1 \cap G_2 \subseteq G_{S_1 \cap S_2}$. Since G_1, G_2 are closed under $\diamond$, $*_\varrho$ for all $\varrho \in S_1 \cap S_2$, the set $G_1 \cap G_2$ is closed under them also. Thus, every two elements of $\mathcal{P}_M$ have the greatest lower bound. Since $\mathcal{P}_M$ is finite and bounded, it follows that $\mathcal{P}_M$ is a lattice.

First, prove the reverse inclusion of (2). Let $(S, G) = (S_1, G_1) \vee (S_2, G_2)$. To prove that $S_1 \mathrm{Pr}(G_2), S_2 \mathrm{Pr}(G_1) \subseteq S$, we take $\alpha = \varrho \alpha'$ where $\varrho \in S_1$ and $(\alpha' - 1, b') \in G_2$. Since $G_2 \subseteq G$, we have $\alpha' \in U_R(S)$. Then, by definition, $\alpha' S \subseteq S$, in particular, $\alpha = \alpha' \varrho \in S$ and $S_1 \mathrm{Pr}(G_2) \subseteq S$. Since $(0, 0) \in G_2 \subseteq G$ and G is closed under $*_\varrho$, we have $(0, 0) *_\varrho (\alpha' - 1, b') = (\varrho\alpha' - \varrho, \varrho b') \in G$. Hence $\alpha \in S$ and $S_1 \cdot \mathrm{Pr}(G_2) \subseteq S$. Repeating this argument for S_2 and G_2, we obtain $S_2 \cdot \mathrm{Pr}(G_1) \subseteq S$. Thus $S' \subseteq S$ and $\langle G_1 \cup G_2 \rangle_{G_{S'}} \subseteq \langle G_1 \cup G_2 \rangle_{G_S} \subseteq G$.

Now, show the inclusions $S \subseteq S'$ and $G \subseteq \langle G_1 \cup G_2 \rangle_{G_{S'}}$. Since $S_1, S_2 \subseteq S'$ and $G_1, G_2 \subseteq \langle G_1 \cup G_2 \rangle_{G_{S'}}$, it suffice to prove that $(S', \langle G_1 \cup G_2 \rangle_{G_{S'}})$ belongs to $\mathcal{P}_M$. By construction, S' is a subring of $\mathrm{Nil}\,(R)$. Let us prove that $\langle G_1 \cup G_2 \rangle_{G_{S'}}$ is a subalgebra of $G_{S'}$, i.e. $G_1, G_2 \subseteq G_{S'}$. Take $\alpha \in \mathrm{Pr}(G_1)$.

For some $b \in M$, we have $(\alpha - 1, b) \in G_1$. By definition, the equality $\alpha S' = \alpha\langle S_1\mathrm{Pr}(G_2) \cup S_2\mathrm{Pr}(G_1)\rangle_R$ holds. Since $\alpha S_1 \subseteq S_1$ and $\mathrm{Pr}(G_1)$ is a group, we have $\alpha S_1\mathrm{Pr}(G_2) \subseteq S_1\mathrm{Pr}(G_2)$ and $\alpha S_2\mathrm{Pr}(G_1) = S_2\alpha\mathrm{Pr}(G_1) = S_2\mathrm{Pr}(G_1)$. So $\alpha(S_1\mathrm{Pr}(G_2) \cup S_2\mathrm{Pr}(G_1)) \subseteq S'$. Therefore if $\alpha\varrho_1, \alpha\varrho_2 \in \langle S_1\mathrm{Pr}(G_2) \cup S_2\mathrm{Pr}(G_1)\rangle_R$ where $\varrho_1, \varrho_2 \in \langle S_1\mathrm{Pr}(G_2) \cup S_2\mathrm{Pr}(G_1)\rangle_R$, then $\alpha(\varrho_1 + \varrho_2), \alpha\varrho_1\varrho_2 \in \langle S_1\mathrm{Pr}(G_2) \cup S_2\mathrm{Pr}(G_1)\rangle_R$. □

The following theorem is the main result of this section.

Theorem 1 *(a) For any primitive algebra* $\mathbf{Q}$*, there is* $(S, G) \in \mathcal{P}_M$ *such that* $f \in \mathbf{Q}$ *if and only if both* $\mathrm{Nil}\,(f) \subseteq S$ *and* $\mathrm{Tr}(f) \in G$.
(b) $[\mathbf{J}_M, \mathbf{LP}_M] \cong \mathcal{P}_M$.

Proof.
Let $\varphi : [\mathbf{J}_M, \mathbf{LP}_M] \to \mathcal{P}_M$ be a map such that $\varphi(\mathbf{Q}) = (\mathrm{Nil}\,(\mathbf{Q}), \mathrm{Tr}(\mathbf{Q}))$ for any primitive algebra $\mathbf{Q}$. Prove that $\varphi(\mathbf{Q}) \in \mathcal{P}_M$ for any $\mathbf{Q}$. If $f \in \mathbf{Q}$, then $g = s_\varrho(x, y, f(z, \dots, z)) = x - \varrho y + \varrho(\sigma(f) + 1)z + \varrho\beta(f) \in \mathbf{Q}$ because $s_\varrho \in \mathbf{Q}$ by Lemma 1. So, $(\sigma(f) + 1) \cdot \mathrm{Nil}\,(\mathbf{Q}) \subseteq \mathrm{Nil}\,(\mathbf{Q})$ and, hence, $\mathrm{Tr}(\mathbf{Q}) \subseteq G_{\mathrm{Nil}\,(\mathbf{Q})}$. By Lemma 3 $\mathrm{Tr}(\mathbf{Q})$ is closed under operations of the algebra $G_{\mathrm{Nil}\,(\mathbf{Q})}$, that is $\varphi(\mathbf{Q}) \in \mathcal{P}_M$. This implies (a).

Now let us prove that φ is an isomorphism required in (b). It follows from Lemma 2 that φ is injective. To show that it is surjective take $(S, G) \in \mathcal{P}_M$. Let $\mathbf{Q}$ be the set of all functions f with the properties $\mathrm{Nil}\,(f) \subseteq S$ and $\mathrm{Tr}(f) \in G$. Take $f = \alpha_1 x_1 + \sum_{i=2}^n \alpha_i x_i + a$, $g = \gamma_1 x_1 + \sum_{i=2}^m \gamma_i x_i + b \in \mathbf{Q}$. Without loss of generality we may assume α_1, γ_1 to be invertible elements. Since $\sigma(f) + 1 \in U_R(S)$ for any $\varrho \in S$, we have $\alpha_1\varrho = (\sigma(f) + 1 - \sum_{i=2}^n \alpha_i)\varrho = \sigma(f)\varrho + \varrho - \sum i = 2^n \alpha_i\varrho \in S$, that is $\alpha_1 \in U_R(S)$. By the same argument, $\gamma_1 \in U_R(S)$ holds. Let $h_1 = f \circ g$, $h_2 = f \circ_j g$, $j \neq 1$. Nilpotent coefficients of h_1 are $\alpha_1\gamma_2, \dots, \alpha_1\gamma_m, \alpha_2, \dots, \alpha_n$ and nilpotent coefficients of h_2 are $\alpha_2, \dots, \alpha_{j-1}$, $\alpha_j\gamma_1, \dots, \alpha_j\gamma_m, \alpha_{j+1}, \dots, \alpha_n$. Since $\alpha_1 \in U_R(S)$, all these coefficients belong to S. Further, $\mathrm{Tr}(h_1) = (\sigma(h_1), \beta(h_1)) = (\sigma(f) + \alpha_1\sigma(g), \beta(f) + \alpha_1\beta(g))$. The element $\varrho = -\sum_{i=2}^n \alpha_i$ belongs to S. Therefore

$$\begin{aligned}
\mathrm{Tr}(h_1) &= (\sigma(f) + \alpha_1\sigma(g), \beta(f) + \alpha_1\beta(g)) = \\
&= (\sigma(f) + (\sigma(f) - \sum_{i=2}^n \alpha_i)\sigma(g) + \sigma(g),
\end{aligned}$$

$$
\begin{aligned}
& \beta(f) + (\sigma(f) - \sum_{i=2}^{n} \alpha_i)\beta(g) + \beta(g)) = \\
= \; & (\sigma(f) + \sigma(f)\sigma(g) + \sigma(g), \beta(f) + \sigma(f)\beta(g) + \beta(g)) *_\varrho \\
& *_\varrho (\sigma(g), \beta(g)) = \\
= \; & ((\sigma(f), \beta(f)) \diamond (\sigma(g), \beta(g))) *_\varrho (\sigma(g), \beta(g)) \in G.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathrm{Tr}(h_2) &= (\sigma(f) + \alpha_j\sigma(g), \beta(f) + \alpha_j\beta(g)) = \\
&= (\sigma(f), \beta(f)) *_{\alpha_j} (\sigma(g), \beta(g)) \in G,
\end{aligned}
$$

because $\alpha_j \in S$. Thus $h_1, h_2 \in \mathbf{Q}$.

Now, let $h_3(x_1, \ldots, x_n) = \Delta_{i,j} f$, $h_4(x_1, \ldots, x_n) = \Delta_{1,j} f$ where $i, j \in \{2, \ldots, n\}$. Since $\sigma(h_3) = \sigma(h_4) = \sigma(f)$ and $\beta(h_3) = \beta(h_4) = \beta(f)$, we have $\mathrm{Tr}(h_3), \mathrm{Tr}(h_4) \in G$. Each element of $\mathrm{Nil}\,(h_3) \cup \mathrm{Nil}\,(h_4)$ is either a coefficient of f or a sum of elements of $\mathrm{Nil}\,(f)$. Hence $\mathrm{Nil}\,(h_3) \cup \mathrm{Nil}\,(h_4) \subseteq S$ and $h_3, h_4 \in \mathbf{Q}$. Evidently, $\mathbf{Q}$ is closed under any permutation of variables and under adding fictitious variables. Thus $\mathbf{Q}$ is an iterative algebra and $\varphi(\mathbf{Q}) = (S, G)$, i.e. φ is surjective.

To complete the proof, note that φ and φ^{-1} are monotone. $\square$

Corollary 1 *For any finite module M, the lattice $[\mathbf{J}_M, \mathbf{LP}_M]$ is finite.*

To illustrate the theorem we draw the lattice of primitive algebras for the ring $\mathbb{Z}_4$ considered as a module over itself. This lattice is shown in Figure 1. There are only two subrings of $\mathrm{Nil}\,(\mathbb{Z}_4)$: $S_1 = \{0\}$ and $S_2 = \{0, 2\}$. We have $G_{S_1} = G_{S_2} = \{(0,0), (0,1), (0,2), (0,3), (2,0), (2,1), (2,2), (2,3)\}$.

We have described the lattice of primitive algebras through a certain set of pairs of special finite algebras. This method was used earlier to describe some other lattices of iterative algebras. For example, in [3] the pairs which consist of a finite ring and a finite module were used very succesfully. In that case, as well as in our one, elements of modelling algebras correspond to suitable features of linear functions (coefficients, their sums, constant terms etc.) This method is based on a statement of the following type: there are only finitely many classes of functions and each function from the same class generates the same algebras. In our case Lemma 2 is such a statement. So, we have a property of functions to be "invertible" in a generalized sense. Therefore, if a function has no invertible coefficient, then it does not enjoy

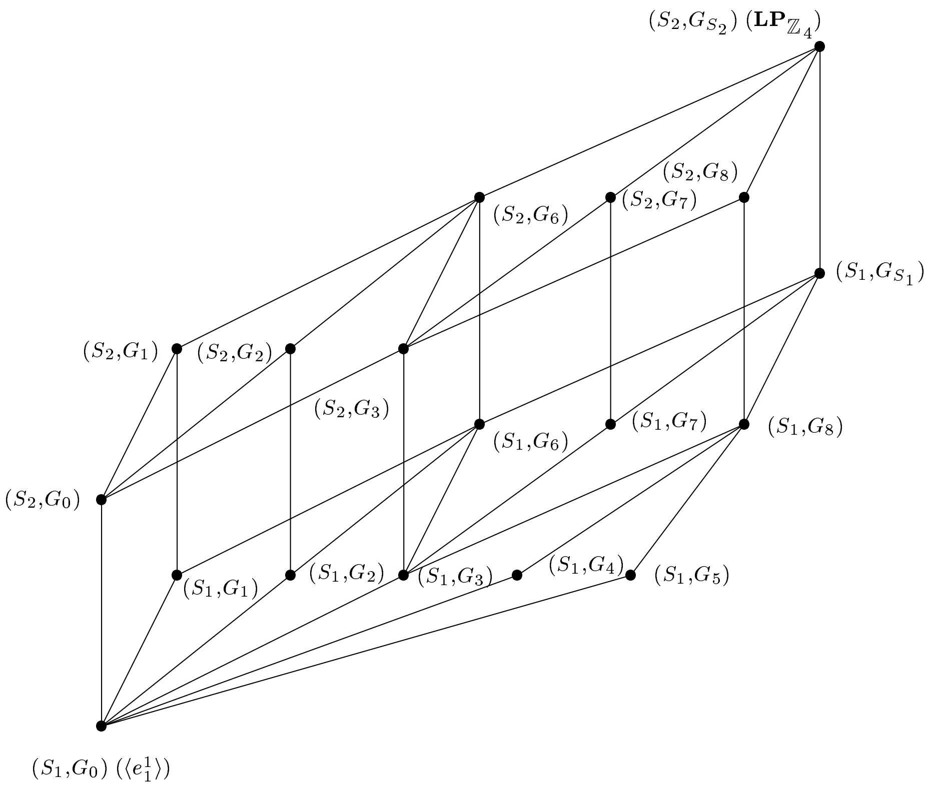

Fig.1

In Figure 1 the following notation is used:
$G_0 = \{(0,0)\}$; thus, (S_1, G_0) corresponds to the algebra of projections, and (S_2, G_0) corresponds to the algebra of all idempotent primitive functions;
$G_1 = \{(0,0),(2,0)\}$, (S_1, G_1) corresponds to the algebra $\langle x, 3x\rangle$ and (S_2, G_1) corresponds to the algebra of primitive functions preserving 0;
$G_2 = \{(0,0),(2,2)\}$ ($\langle x, 3x+2\rangle$ is a preimage of (S_1, G_2)); $G_3 = \{(0,0),(0,2)\}$ ((S_1, G_3) corresponds to $\langle x, x+2\rangle$);
$G_4 = \{(0,0),(2,1)\}$; $G_5 = \{(0,0),(2,3)\}$ (sets G_4 and G_5 are not subalgebras of G_{S_2}); $G_6 = \{(0,0),(2,0),(2,2),(0,2)\}$; $G_7 = \{(0,0),(0,1),(0,2),(0,3)\}$ (the corresponding algebra is $\langle x, x+1, x+2, x+3\rangle$);
$G_8 = \{(0,0),(0,2),(2,1),(2,3)\}$;
finally, (S_1, G_{S_1}) corresponds to the algebra of all unary primitive functions and (S_2, G_{S_2}) corresponds to $\mathbf{LP}_{\mathbb{Z}_4}$.

this property. Thus the "invertibility" of a function implies that it has an invertible coefficient. Hence, it is very reliable that one can't study nilpotent algebras in this way.

If an algebra contains a function with an invertible coefficient, then by Propositions 2, 3, 5 from [1] it contains either $x - y + z$ or $\varepsilon x + (1-\varepsilon)y$ where $\varepsilon \in E(R)\backslash\{0,1\}$, or a primitive function. The first possibility have been considered in [3]. Theorem 1 from [1] covers the second possibility and the third one is partially studied in this paper. Thus the applicability of this method in our case seems to be exhausted.

§2. Algebras which consist of primitive and nilpotent functions

In this section we deal with the lattice of subalgebras of the algebra $\mathbf{NP}_M = \mathbf{LP}_M \vee \mathbf{N}_M = \mathbf{LP}_M \cup \mathbf{N}_M$. According to Theorem 1 of [1] this lattice is infinite if and only if Sub $\mathbf{N}_M$ is infinite and, consequently, if and only if R contains a nilpotent element. For a primitive algebra $\mathbf{Q}$, let us denote by $\mathrm{LNP}(\mathbf{Q})$ the special set of subalgebras of $\mathbf{NP}_M$. This set consists of all subalgebras F such that $\mathrm{F} \cap \mathbf{LP}_M = \mathbf{Q}$. The set $\mathrm{LNP}(\mathbf{Q})$ is non-empty for any $\mathbf{Q}$ since $\mathbf{Q} \in \mathrm{LNP}(\mathbf{Q})$. It will be proved that $\mathrm{LNP}(\mathbf{Q})$ forms an interval of Sub $\mathbf{NP}_M$ and is isomorphic to a certain sublattice of Sub $\mathbf{N}_M$. In the first subsection we describe the lattice operations of Sub $\mathbf{NP}_M$ up to an interval of the form $\mathrm{LNP}(\mathbf{Q})$. More precisely, given any $\mathrm{F}_1, \mathrm{F}_2 \in$ Sub $\mathbf{NP}_M$ we determine primitive algebras $\mathbf{Q}_1, \mathbf{Q}_2$ such that $\mathrm{F}_1 \vee \mathrm{F}_2 \in \mathrm{LNP}(\mathbf{Q}_1)$, $\mathrm{F}_1 \wedge \mathrm{F}_2 \in \mathrm{LNP}(\mathbf{Q}_2)$. In the second one we formulate and prove some results characterizing primitive algebras $\mathbf{Q}$ with a finite interval $\mathrm{LNP}(\mathbf{Q})$, and describe $\mathrm{LNP}(\mathbf{Q})$ in this case.

2.1. Operations of Sub $\mathbf{NP}_M$

Every algebra from $\mathbf{NP}_M$ is the (set-theoretical) union of a primitive algebra and a nilpotent one. Thus Sub $\mathbf{NP}_M$ is a subset of the direct product of the lattices Sub $\mathbf{N}_M$ and $[\mathbf{J}_M, \mathbf{LP}_M]$, but is not a subdirect product of them. Lemmas 7, 8, and Proposition 2 give some information on the nilpotent components of algebras from $\mathrm{LNP}(\mathbf{Q})$ for a fixed $\mathbf{Q}$.

We need two lemmas which state trivial properties of unary operations of an iterative algebra.

Lemma 5 *For any f, g, $\mathrm{ar} f = n$, $\mathrm{ar} g = m$, the following identities are valid*

$$\begin{aligned} f \circ (\Delta g) &= \Delta(f \circ g); \\ f \circ (\nabla g) &= \nabla(f \circ g); \\ (\Delta f) \circ g &= f(g(x_1, \dots, x_m), g(x_1, \dots, x_m), x_{m+1}, \dots, x_{m+n-2}) = \\ &= \Delta_{m,m+1} \dots \Delta_{1,m+1}((f \circ_2 g) \circ g); \\ (\nabla f) \circ g &= \underbrace{\nabla \dots \nabla}_{m \text{ times}} f. \end{aligned}$$

For any permutations π_1 of $\underline{n}$ and π_2 of $\underline{m}$, there exist permutatins π_1', π_2' of $\underline{m+n-1}$ such that

$$\begin{aligned} f \circ \pi_2 g &= \pi_2'(f \circ g); \\ (\pi_1 f) \circ_j g &= \pi_1'(f \circ_{\pi_1^{-1}(j)} g). \end{aligned}$$

Lemma 6 *Let $f \in \mathbf{L}_M$ and $\theta \in \{\tau, \xi, \Delta, \nabla\}$. Then $\mathrm{Tr}(\theta f) = \mathrm{Tr}(f)$ and the coefficients of the function θf are equal either to coefficients of f or to sums of coefficients of f. Moreover, if f is primitive and α is the invertible element of f, then the invertible element of the function θf either equal to α or belongs to $\alpha + \mathrm{Nil}\,(f)$.*

If a function f is of the form $f(x_1, \dots, x_n) = \sum_{i=1}^n \alpha_i x_i + a$, then α_1 will be called the first coefficient of f. For a primitive algebra $\mathbf{Q}$, put

$$\begin{aligned} \mathbf{N}(\mathbf{Q}) = \{f \in \mathbf{N}_M \mid &\text{ for all } g \in \mathbf{Q}, \text{ if the first coefficient} \\ &\text{of } g \text{ is nilpotent, then } g \circ f \in \mathbf{Q}\}. \end{aligned}$$

Also, let $\mathrm{LN}(\mathbf{Q}) = \{\mathbf{N} \in \mathrm{Sub}\,\mathbf{NP}_M \mid \mathbf{LP}_M \cap (\mathbf{N} \vee \mathbf{Q}) = \mathbf{Q}\} \subseteq \mathrm{Sub}\,\mathbf{N}_M$.

Lemma 7 *Let $\mathbf{Q}$ be a primitive algebra. The set $\mathrm{LN}(\mathbf{Q})$ is a principal ideal of $\mathrm{Sub}\,\mathbf{N}_M$ generated by $\mathbf{N}(\mathbf{Q})$.*

Proof.

If $\mathbf{N}_1, \mathbf{N}_2 \in \mathrm{Sub}\,\mathbf{N}_M$, $\mathbf{N}_1 \subseteq \mathbf{N}_2$ and $\mathbf{LP}_M \cap (\mathbf{N}_2 \vee \mathbf{Q}) = \mathbf{Q}$, then $\mathbf{LP}_M \cap (\mathbf{N}_1 \vee \mathbf{Q}) = \mathbf{Q}$. To show that each element of $\mathrm{LN}(\mathbf{Q})$ is a subalgebra of $\mathbf{N}(\mathbf{Q})$ take $\mathbf{N} \in \mathrm{LN}(\mathbf{Q})$. If $f \in \mathbf{N}$, $g \in \mathbf{Q}$ and the first coefficient of g is nilpotent, then $g \circ f$ is a primitive function and $g \circ f \in \mathbf{Q}$. Therefore, $f \in \mathbf{N}(\mathbf{Q})$, i.e. $\mathbf{N} \subseteq \mathbf{N}(\mathbf{Q})$.

Let us prove that $\mathbf{N}(\mathbf{Q})$ is an iterative algebra. By Lemma 5 $\mathbf{N}(\mathbf{Q})$ is closed under permutations of variables, their identification, and adding a fictitious variable. If $f, h \in \mathbf{N}(\mathbf{Q})$, $g \in \mathbf{Q}$ and the first coefficient of g is nilpotent, then $g \circ (f \circ h) = (g \circ f) \circ h \in \mathbf{Q}$. So, $\mathbf{N}(\mathbf{Q})$ is closed under substitution, i.e. $\mathbf{N}(\mathbf{Q})$ is a subalgebra of $\mathbf{N}_M$.

To show that $\mathbf{N}(\mathbf{Q}) \in \mathbb{L}\mathrm{N}(\mathbf{Q})$ we prove the equality $\mathbf{N}(\mathbf{Q}) \vee \mathbf{Q} = \mathbf{N}(\mathbf{Q}) \cup \mathbf{Q}$. Since $\mathbf{N}(\mathbf{Q})$ and $\mathbf{Q}$ are closed under the operations $\tau, \xi, \nabla, \Delta$, the set $\mathbf{N}(\mathbf{Q}) \cup \mathbf{Q}$ is closed under them, as well. Let $f = h_1 \circ h_2$ where $h_1, h_2 \in \mathbf{N}(\mathbf{Q}) \cup \mathbf{Q}$. If $h_1, h_2 \in \mathbf{N}(\mathbf{Q})$ or $h_1, h_2 \in \mathbf{Q}$, then $f \in \mathbf{N}(\mathbf{Q}) \cup \mathbf{Q}$. To check the other cases take $g \in \mathbf{Q}$ such that the first coefficient of g is nilpotent. We have $g \circ f = g \circ (h_1 \circ h_2) = (g \circ h_1) \circ h_2$. If $h_1 \in \mathbf{Q}$, then $g \circ h_1 \in \mathbf{Q}$ because $g \in \mathbf{Q}$. If $h_1 \in \mathbf{N}(\mathbf{Q})$, then $g \circ h_1 \in \mathbf{Q}$ by definition of $\mathbf{N}(\mathbf{Q})$. In both cases the first coefficient of $g \circ h_1$ is nilpotent. Consequently, $(g \circ h_1) \circ h_2 \in \mathbf{Q}$ and so either $f \in \mathbf{N}(\mathbf{Q})$ or f is a primitive function. Thus $\mathbf{N}(\mathbf{Q}) \vee \mathbf{Q} = \mathbf{N}(\mathbf{Q}) \cup \mathbf{Q}$ and the proof is complete. $\square$

For $f \in \mathbf{L}_M$ let $\mathrm{K}(f)$ denote the set of coefficients of f and for $\mathbf{C} \in \mathrm{Sub}\, \mathbf{L}_M$, let $\mathrm{K}(C) = \bigcup_{f \in \mathbf{C}} \mathrm{K}(f)$.

Lemma 8 $\mathbf{N}(\mathbf{Q}) = \{f \in \mathbf{N}_M \mid \mathrm{K}(f)\mathrm{Nil}\,(\mathbf{Q}) \subseteq \mathrm{Nil}\,(\mathbf{Q})$ *and* $\mathrm{Nil}\,(\mathbf{Q}) \cdot \mathrm{Tr}(f) \subseteq \mathrm{Tr}(\mathbf{Q})\}$. *(3)*

Proof.

If $f \in \mathbf{N}(\mathbf{Q})$, then $\mathrm{K}(s_\varrho \circ_3 f) = \{1, -\varrho\} \cup \varrho \cdot \mathrm{K}(f)$ and $\mathrm{Tr}(s_\varrho \circ_3 f) = \mathrm{Tr}(s_\varrho) *_\varrho \mathrm{Tr}(f) = \varrho \cdot \mathrm{Tr}(f)$. Hence, f belongs to the set in the right side of (3). Let us show the reverse inclusion. Since $\mathbf{Q}$ is completely determined by $\mathrm{Nil}\,(\mathbf{Q})$ and $\mathrm{Tr}(\mathbf{Q})$, we have

$$\begin{aligned} \mathbf{N}(\mathbf{Q}) \;=\; & \{f \in \mathbf{N}_M \mid \mathrm{K}(f)\mathrm{Nil}\,(\mathbf{Q}) \subseteq \mathrm{Nil}\,(\mathbf{Q}),\; \mathrm{Tr}(g) *_\varrho \mathrm{Tr}(f) \in \mathrm{Tr}(\mathbf{Q}) \\ & \text{for all } \varrho \in \mathrm{Nil}\,(\mathbf{Q}),\, g \in \mathbf{Q}\}. \end{aligned}$$

Suppose $\varrho \in \mathrm{Nil}\,(\mathbf{Q})$, $h \in \mathbf{Q}$, and g belongs to the set $\{f \in \mathbf{N}_M \mid \mathrm{K}(f)\mathrm{Nil}\,(\mathbf{Q}) \subseteq \mathrm{Nil}\,(\mathbf{Q}),\; \mathrm{Nil}\,(\mathbf{Q}) \cdot \mathrm{Tr}(f) \subseteq \mathrm{Tr}(\mathbf{Q})\}$. Then

$$\begin{aligned} \mathrm{Tr}(h) *_\varrho \mathrm{Tr}(g) &= (\sigma(h) + \varrho\sigma(g), \beta(h) + \varrho\beta(g)) = \\ &= (\varrho\sigma(g) + \varrho\sigma(g)\sigma(h) + \sigma(h) - \varrho\sigma(g)\sigma(h), \\ &\quad \varrho\beta(g) + \varrho\sigma(g)\beta(h) + \beta(h) - \varrho\sigma(g)\beta(h)) = \end{aligned}$$

$$\begin{aligned} &= ((\varrho\sigma(g), \varrho\beta(g)) \diamond (\sigma(h), \beta(h))) *_{-\varrho\sigma(g)} (\sigma(h), \beta(h)) = \\ &= (\varrho \mathrm{Tr}(g) \diamond \mathrm{Tr}(h)) *_{-\varrho\sigma(h)} \mathrm{Tr}(h) \in \mathrm{Tr}(\mathbf{Q}) \end{aligned}$$

because $-\varrho\sigma(h) \in -\varrho(U_R(S) - 1) \in S$. Thus $g \in \mathbf{N}(\mathbf{Q})$ and the lemma is proved. $\square$

For a primitive algebra $\mathbf{Q}$, we will denote the set $\{\mathbf{N} \in \mathbb{LN}(\mathbf{Q}) \mid (\mathbf{N} \vee \mathbf{Q}) \cap \mathbf{N}_M = \mathbf{N}\}$ by $\mathbb{LNN}(\mathbf{Q})$. Obviously, the mapping taking $\mathbf{N} \in \mathbb{LNN}(\mathbf{Q})$ to $\mathbf{N} \cup \mathbf{Q} \in \mathbb{LNP}(\mathbf{Q})$ is an isomorphism of the lattices $\mathbb{LNN}(\mathbf{Q})$ and $\mathbb{LNP}(\mathbf{Q})$. Note, that a nilpotent algebra $\mathbf{N}$ belongs to $\mathbb{LNN}(\mathbf{Q})$ if and only if $\mathbf{N} \vee \mathbf{Q} = \mathbf{N} \cup \mathbf{Q}$. If $F \subseteq \mathbf{P}_A$ and $f \in \langle F \rangle$, then the function f can be represented by a term of type $\{\circ, \tau, \xi, \nabla, \Delta\}$ of iterative algebra. Below we often use induction on the leth of such a term.

Proposition 2 *For any primitive algebra* $\mathbf{Q}$*, the set* $\mathbb{LNN}(\mathbf{Q})$ *is a sublattice of* $\mathrm{Sub}\, \mathbf{N}_M$.

Proof.
Suppose $\mathbf{N}_1, \mathbf{N}_2 \in \mathbb{LNN}(\mathbf{Q})$. Then

$$(\mathbf{Q} \vee \mathbf{N}_1) \wedge (\mathbf{Q} \vee \mathbf{N}_2) = (\mathbf{Q} \cup \mathbf{N}_1) \cap (\mathbf{Q} \cup \mathbf{N}_2) = \mathbf{Q} \cup (\mathbf{N}_1 \cap \mathbf{N}_2) = \mathbf{Q} \vee (\mathbf{N}_1 \wedge \mathbf{N}_2),$$

i.e. $\mathbf{N}_1 \wedge \mathbf{N}_2 \in \mathbf{N}(\mathbf{Q})$. Let us check that $\mathbf{N}_1 \vee \mathbf{N}_2 \in \mathbb{LNN}(\mathbf{Q})$, that is $(\mathbf{N}_1 \vee \mathbf{N}_2 \vee \mathbf{Q}) \cap \mathbf{N}_M = \mathbf{N}_1 \vee \mathbf{N}_2$ holds. Obviously, $(\mathbf{N}_1 \vee \mathbf{N}_2 \vee \mathbf{Q}) \cap \mathbf{N}_M \supseteq \mathbf{N}_1 \vee \mathbf{N}_2$. So we should prove the reverse inclusion. Take $f \in \langle \mathbf{N}_1 \cup \mathbf{N}_2 \cup \mathbf{Q} \rangle$ and prove by induction on the length of the term which represents f that $f \in \mathbf{N}_M$ implies $f \in \mathbf{N}_1 \vee \mathbf{N}_2$. The case $f \in \mathbf{N}_1 \cup \mathbf{N}_M$ is obvious. Further, if $f = \theta(f')$ where $\theta \in \{\tau, \xi, \nabla, \Delta\}$ and $f' \in \mathbf{N}_1 \vee \mathbf{N}_2$, then $f \in \mathbf{N}_1 \vee \mathbf{N}_2$. (Since $f \in \mathbf{N}_1 \vee \mathbf{N}_2$, we have $f' \notin \mathbf{Q}$.) Now, let $f = f_1 \circ f_2$ where $f_1, f_2 \in (\mathbf{N}_1 \vee \mathbf{N}_2) \cup \mathbf{Q}$. If $f_1, f_2 \in \mathbf{Q}$ or $f_1, f_2 \in \mathbf{N}_1 \vee \mathbf{N}_2$, then, obviously, $f \in (\mathbf{N}_1 \vee \mathbf{N}_2) \cup \mathbf{Q}$. If f_1 is primitive and the first coefficient of f_1 is a nilpotent element, and $f_2 \in \mathbf{N}_1 \vee \mathbf{N}_2$, then $f_1 \circ f_2 \in \mathbf{Q}$, since $f_2 \in \mathbf{N}(\mathbf{Q})$. So, we have to consider the following two cases.

A. $f = h \circ g$ where $h \in \mathbf{Q}$, $g \in \mathbf{N}_1 \vee \mathbf{N}_2$ and the first coefficient of h is invertible.

B. $f = g \circ_j h$ where $g \in (\mathbf{N}_1 \vee \mathbf{N}_2)^{(n)}$, $h \in \mathbf{Q}$ and $j \in \underline{n}$.

By induction on the length of a term which represents g we prove that $f \in \mathbf{N}_1 \vee \mathbf{N}_2$ in both cases. If $g \in \mathbf{N}_1 \cup \mathbf{N}_2$, say $g \in \mathbf{N}_1$, then $f \in \mathbf{N}_1 \vee \mathbf{Q} =$

$\mathbf{N}_1 \cup \mathbf{Q}$. Consequently, $f \in \mathbf{N}_1$. Lemma 6 implies that if $g = \theta(g')$ where $\theta \in \{\tau, \xi, \Delta, \nabla\}$ and $h \circ g' \in \mathbf{N}_1 \vee \mathbf{N}_2$, then $f \in \mathbf{N}_1 \vee \mathbf{N}_2$ in both cases. Therefore, assume that $g = g_1 \circ g_2$ where $h \circ g_1 \in \mathbf{N}_1 \vee \mathbf{N}_2$ (respectively, $g_1 \circ_l h,\quad g_2 \circ_l h \in \mathbf{N}_1 \vee \mathbf{N}_2$ for any l).

A. Since $h \circ g_1 \in \mathbf{N}_1 \vee \mathbf{N}_2$ by inductive hypothesis, we have $f = h \circ g = h \circ (g_1 \circ g_2) = (h \circ g_1) \circ g_2 \in \mathbf{N}_1 \vee \mathbf{N}_2$.

B. Suppose that $\mathrm{ar} g_1 = n_1$, $\mathrm{ar} g_2 = n_2$. If $j \in \underline{n_2}$, then $f = g \circ_j h = (g_1 \circ g_2) \circ_j h = g_1 \circ (g_2 \circ_j h) \in \mathbf{N}_1 \vee \mathbf{N}_2$. Further, if $j \in \{n_2+1, \ldots, n_1+n_2+1\}$, then $f = g \circ_j h = (g_1 \circ g_2) \circ_j h = (g_1 \circ_{j-n_2+1} h) \circ g_2 \in \mathbf{N}_1 \vee \mathbf{N}_2$.

So, in both cases the function f belongs to $\mathbf{N}_1 \vee \mathbf{N}_2$. Thus, $(\mathbf{N}_1 \vee \mathbf{N}_2 \vee \mathbf{Q}) \cap \mathbf{N}_M = \mathbf{N}_1 \vee \mathbf{N}_2$ and the proof is complete. $\square$

To describe the operations of Sub $\mathbf{NP}_M$ we need to know the structure of the lattice of nilpotent algebras. However, this lattice seems to be rather complicated. So, it is very difficult to study the operations of Sub $\mathbf{NP}_M$. Let $\mathrm{F}_1, \mathrm{F}_2 \in$ Sub $\mathbf{NP}_M$. The following statement determines a primitive algebra $\mathbf{Q}$ such that $\mathrm{F}_1 \vee \mathrm{F}_2 \in \mathbb{LNP}(\mathbf{Q})$, using the sets of coefficients and traces of algebras $\mathrm{F}_1, \mathrm{F}_2$.

Proposition 3 *Let* $\mathrm{F}_1, \mathrm{F}_2 \in$ Sub $\mathbf{NP}_M$ *and* $\mathrm{F}_i = \mathbf{Q}_i \cup \mathbf{N}_i$ *where* $\mathbf{Q}_i = \mathrm{F}_i \cap \mathbf{LP}_M \in \mathrm{Sub}\, \mathbf{LP}_M$, $\mathbf{N}_i = \mathrm{F}_i \cap \mathbf{N}_M \in \mathrm{Sub}\, \mathbf{N}_M$, $i = 1, 2$. *Put* $S_i = \mathrm{Nil}\,(\mathbf{Q}_i)$, $G_i = \mathrm{Tr}(\mathbf{Q}_i)$ *where* $i = 1, 2$. *Then for* $\mathbf{Q}' = (\mathrm{F}_1 \vee \mathrm{F}_2) \cap \mathbf{LP}_M$, *we have*

$$\mathrm{Nil}\,(\mathbf{Q}') = \langle \mathrm{Nil}\,(\mathbf{Q}_1 \vee \mathbf{Q}_2) \cup S_2 \cdot \mathrm{K}(\mathbf{N}_1) \cup S_1 \cdot \mathrm{K}(\mathbf{N}_2) \rangle_R \ \ (= S); \qquad (4)$$

$$\mathrm{Tr}(\mathbf{Q}') = \langle G_1 \cup G_2 \cup S_1 \cdot \mathrm{Tr}(\mathbf{N}_2) \cup S_2 \cdot \mathrm{Tr}(\mathbf{N}_1) \rangle_{G_S} \ \ (= G). \qquad (5)$$

Proof.

By Proposition 1 the equality

$$S = \langle S_2 \cdot \mathrm{Pr}(G_1) \cup S_2 \cdot \mathrm{K}(\mathbf{N}_1) \cup S_1 \cdot \mathrm{Pr}(G_2) \cup S_1 \cdot \mathrm{K}(\mathbf{N}_2) \rangle_R$$

is valid. To prove $\mathrm{Nil}\,(\mathbf{Q}') \supseteq S$ let us take any $\varrho \in S_2 \cdot \mathrm{Pr}(G_1) \cup S_1 \cdot \mathrm{Pr}(G_2) \cup S_2 \cdot \mathrm{K}(\mathbf{N}_1) \cup S_1 \cdot \mathrm{K}(\mathbf{N}_2)$ and show that s_ϱ belongs to $\mathbf{Q}'$. If $\varrho \in S_2 \cdot \mathrm{Pr}(G_1) \cup S_1 \cdot \mathrm{Pr}(G_2)$, then $s_\varrho \in \mathbf{Q}'$ by Theorem 1 and equality (2) of Proposition 1. Suppose $\varrho = \varrho'\varrho''$ where $\varrho' \in S_2$, $\varrho'' \in \mathrm{K}(\mathbf{N}_1)$. Then $\alpha x + \varrho'' y + c \in \mathbf{N}_1$ and $(1-\varrho')x + \varrho' y \in \mathbf{Q}_2$ for a certain $\alpha \in R$ and $c \in M$. Consequently, $(1-\varrho')x + \varrho'(\alpha x + \varrho'' y + c) = (1 - \varrho' + \varrho'\alpha)x + \varrho''\varrho' y + \varrho' c \in \mathbf{Q}'$, i.e. $\varrho \in S$. Repeating this computation, one can show that $S_1 \mathrm{K}(\mathbf{N}_2) \subseteq S$.

Let us prove that $\mathrm{Tr}(\mathbf{Q}') \supseteq G$. The function $(\alpha+1)x+b$ belongs to $\mathbf{Q}'$ for any pair $(\alpha, b) \in S_1 \cdot \mathrm{Tr}(\mathbf{N}_2) \cup S_2 \cdot \mathrm{Tr}(\mathbf{N}_1)$. Indeed, if $(\alpha, b) = (\varrho\alpha_1, \varrho b_1)$ where $\varrho \in S_2$ and $(\alpha_1, b_1) \in \mathrm{Tr}(\mathbf{N}_1)$, then $s_\varrho \in \mathbf{Q}_2$ and $(\alpha_1+1)x+b_1 \in \mathbf{N}_1$. Hence, $x - \varrho x + \varrho((\alpha_1+1)x+b_1) = (\varrho\alpha_1+1)x+\varrho b_1 = (\alpha+1)x+b \in \mathbf{Q}'$. If $\varrho \in S_1$ and $(\alpha_1, b_1) \in \mathrm{Tr}(\mathbf{N}_2)$, then the argument is similar.

Inside the proof we prove an auxiliary claim.

Claim 1 $S \cdot \langle \mathrm{Pr}(G_1) \cup S_1 \cup \mathrm{Pr}(G_2) \cup S_2 \cup \mathrm{K}(\mathbf{N}_1) \cup \mathrm{K}(\mathbf{N}_2) \rangle \subseteq S$.

It must be shown that, for any $\varrho_1 \in S$ and any $\varrho_2 \in \mathrm{Pr}(G_1) \cup S_1 \cup \mathrm{Pr}(G_2) \cup S_2 \cup \mathrm{K}(\mathbf{N}_1) \cup \mathrm{K}(\mathbf{N}_2)$, the product $\varrho_1\varrho_2$ belongs to S. It is enough to take ϱ_1 from the generating set $S_2 \cdot \mathrm{Pr}(G_1) \cup S_2 \cdot \mathrm{K}(\mathbf{N}_1) \cup S_1 \cdot \mathrm{Pr}(G_2) \cup S_1 \cdot \mathrm{K}(\mathbf{N}_2)$ of S. We have:
$S_2 \cdot \mathrm{Pr}(G_1) \cdot \mathrm{Pr}(G_1) \subseteq S_2 \cdot \mathrm{Pr}(G_1) S_2$ since $\mathrm{Pr}(G_1)$ is a group;
$S_2 \cdot \mathrm{Pr}(G_1) \cdot \mathrm{Pr}(G_2) \subseteq S_2 \cdot \mathrm{Pr}(G_1)$ since $\mathrm{Pr}(G_2) \subseteq U_R(S_2)$;
$S_2 \cdot \mathrm{Pr}(G_1) \cdot \mathrm{K}(\mathbf{N}_1) \subseteq S_2 \cdot \mathrm{K}(\mathbf{N}_1)$ since $\mathrm{Pr}(G_1) \cdot \mathrm{K}(\mathbf{N}_1) \subseteq \mathrm{K}(\mathbf{N}_1)$;
$S_2 \cdot \mathrm{Pr}(G_1) \cdot \mathrm{K}(\mathbf{N}_2) \subseteq S_2 \cdot \mathrm{Pr}(G_1)$ since $S_2 \cdot \mathrm{K}(\mathbf{N}_2) \subseteq S_2$;
$\mathrm{Pr}(G_1) \cdot S_2 \cdot S_1 \subseteq S_2 \cdot \mathrm{Pr}(G_1) \cdot S_1 \cdot \mathrm{Pr}(G_2)$ since $1 \in \mathrm{Pr}(G_2)$;
$S_2 \cdot \mathrm{Pr}(G_1) \cdot S_2 \subseteq S_2 \cdot \mathrm{Pr}(G_1)$.
For $S_1 \cdot \mathrm{Pr}(G_2)$ the proof is similar. By the same reasons:
$S_2 \cdot \mathrm{K}(\mathbf{N}_1) \cdot \mathrm{Pr}(G_1) \subseteq S_2 \cdot \mathrm{K}(\mathbf{N}_1)$; $S_2 \cdot \mathrm{K}(\mathbf{N}_1) \cdot S_1 \subseteq S_1 S_2$;
$S_2 \cdot \mathrm{K}(\mathbf{N}_1) \mathrm{Pr}(G_2) \subseteq S_2 \cdot \mathrm{K}(\mathbf{N}_1)$; $S_2 \cdot \mathrm{K}(\mathbf{N}_1) \cdot S_2 \subseteq S_2 \cdot \mathrm{K}(\mathbf{N}_1)$;
$S_2 \cdot \mathrm{K}(\mathbf{N}_1) \cdot \mathrm{K}(\mathbf{N}_1) \subseteq S_2 \cdot \mathrm{K}(\mathbf{N}_1)$, since $\mathrm{K}(\mathbf{N}_1)$ is closed under multiplication;
$S_2 \cdot \mathrm{K}(\mathbf{N}_1) \cdot \mathrm{K}(\mathbf{N}_2) \subseteq S_2 \cdot \mathrm{K}(\mathbf{N}_1)$.
For $\mathrm{K}(\mathbf{N}_2) S_1$ the argument is similar. The claim is proved.

Now, let us show that $\mathrm{Nil}\,(\mathbf{Q}') \subseteq S$. Put $\mathbf{N}' = (\mathrm{F}_1 \vee \mathrm{F}_2) \cap \mathbf{N}_M$. We have $\mathbf{N}' \subseteq \mathrm{F}_1 \vee \mathrm{F}_2 = \mathbf{Q}' \vee \mathbf{N}'$. Since $\mathbf{N}' \subseteq \langle \mathbf{N}_1 \cup \mathbf{N}_2 \cup \mathbf{Q}_1 \cup \mathbf{Q}_2 \rangle$, all elements from $\mathrm{K}(\mathbf{N}')$ belong to the ring generated by $\mathrm{K}(\mathbf{N}_1) \cup \mathrm{K}(\mathbf{N}_2) \cup \mathrm{K}(\mathbf{Q}_1) \cup \mathrm{K}(\mathbf{Q}_2)$. Consequently, $\mathrm{K}(\mathbf{N}') \subseteq \langle \mathrm{Pr}(G_1) \cup S_1 \cup \mathrm{Pr}(G_2) \cup S_2 \cup \mathrm{K}(\mathbf{N}_1) \cup \mathrm{K}(\mathbf{N}_2) \rangle$. Now we should prove that $\mathrm{Nil}\,(f) \subseteq S$ and that the invertible coefficient of f belongs to $U_R(S)$ for any $f \in \mathbf{Q}'$. The proof is by induction on the length of a term which represents f. If $f \in \mathbf{Q}_1 \cup \mathbf{Q}_2$, then, obviously, $\mathrm{Nil}\,(f) \subseteq S$ and by Claim 1 the invertible coefficient of f belongs to $U_R(S)$. If $f = \theta(f')$ where $\mathrm{Nil}\,(f') \in S$ and $\theta \in \{\tau, \xi, \nabla, \Delta\}$, then $\mathrm{Nil}\,(f) \in S$. The invertible coefficient of f either is equal to the invertible coefficient α of f' or belongs to the set $\alpha + S$ (when $\theta = \Delta$). In both cases it belongs to $U_R(S)$.

If $f = h \circ g$, we have to consider two cases. (Since f is primitive, h must be primitive, too.)

Case 1. $h, g \in \mathbf{Q}'$. By inductive hipothesis, $\mathrm{Nil}\,(h), \mathrm{Nil}\,(g) \subseteq S$ and the invertible coefficients α of h and β of g belong to $U_R(S)$. Therefore, $\mathrm{Nil}\,(f) \subseteq \mathrm{Nil}\,(h) \cup \mathrm{Nil}\,(g) \cup \beta \cdot \mathrm{Nil}\,(h) \cup \alpha \cdot \mathrm{Nil}\,(g) \subseteq S$. The invertible coefficient of f belongs to $U_R(S)$ by Lemma 6.

Case 2. $h \in \mathbf{Q}'$, $g \in \mathbf{N}'$ and the first coefficient ϱ of h is a nilpotent element. The invertible coefficients of f and g are equal. Finally, $\varrho \mathrm{K}(g) \subseteq S$ by Claim 1.

To prove that $\mathrm{Tr}(\mathbf{Q}') \subseteq G$ we take $f \in \mathbf{Q}'$ and show that $\mathrm{Tr}(f) \in G$. We will prove this by induction on the length of a term which represents f. In the cases $f \in \mathbf{Q}_1 \cup \mathbf{Q}_2 \cup \mathbf{N}_1 \cup \mathbf{N}_2$ and $f = \theta(f')$ where $\theta \in \{\tau, \xi, \Delta, \nabla\}$ and $\mathrm{Tr}(f') \in G$, the proof is obvious. Suppose that $f = g \circ h$. Then $g \in \mathbf{LP}_M$ what implies $\mathrm{Tr}(g) \in G$. Further, if $h \in \mathbf{LP}_M$, then $\mathrm{Tr}(h) \in G$. So, $\mathrm{Tr}(f) \in G$. Finally, if $h \in \mathbf{N}_M$, then the condition $\mathrm{Tr}(f) \in G$ is a consequence of the following claim.

Claim **2** *For any $g \in \mathbf{Q}'$, $h \in \mathbf{N}'$, $\varrho \in S$, the condition $\mathrm{Tr}(g) \in G$ implies $\mathrm{Tr}(g) *_\varrho \mathrm{Tr}(h) \in G$.*

We prove the statement by induction on the length of a term which represents h. First, let $h \in \mathbf{N}_1 \cup \mathbf{N}_2$. Then

$$
\begin{aligned}
&\mathrm{Tr}(g) *_\varrho \mathrm{Tr}(h) \\
&= (\sigma(g) + \varrho\sigma(h), \beta(g) + \varrho\beta(h)) = \\
&= (\varrho\sigma(h) + \varrho\sigma(h)\sigma(g) + \sigma(g) - \varrho\sigma(h)\sigma(g), \\
&\quad\ \varrho\beta(h) + \varrho\sigma(h)\beta(g) + \beta(g) - \varrho\sigma(h)\beta(g)) = \\
&= (\varrho\sigma(h) + \varrho\sigma(h)\sigma(g) + \sigma(g), \varrho\beta(h) + \varrho\sigma(h)\beta(g) + \beta(g)) *_{-\varrho\sigma(h)} \\
&\qquad *_{-\varrho\sigma(h)} (\sigma(g), \beta(g)) = \\
&= ((\varrho\sigma(h), \varrho\beta(h)) \diamond (\sigma(g), \beta(g)) *_{-\varrho\sigma(h)} \mathrm{Tr}(g) = \\
&= (\varrho\mathrm{Tr}(h) \diamond \mathrm{Tr}(g)) *_{-\varrho\sigma(h)} \mathrm{Tr}(g).
\end{aligned}
$$

The trace $\varrho\mathrm{Tr}(h)$ belongs to G. Indeed, if $\varrho \in S_1 \cup S_2$, then $\varrho\mathrm{Tr}(h) \in G$ by definition of G and by Lemma 8. Further, let $\varrho_1 \mathrm{Tr}(h), \varrho_2 \mathrm{Tr}(h) \in G$. We have

$$
\begin{aligned}
\varrho_1\varrho_2\mathrm{Tr}(h) &= (0,0) *_{\varrho_1} \varrho_2\mathrm{Tr}(h) \quad \text{and} \\
(\varrho_1 + \varrho_2)\mathrm{Tr}(h) &= (\varrho_1\mathrm{Tr}(h) \diamond \varrho_2\mathrm{Tr}(h)) *_{-\varrho_1} \varrho_2\mathrm{Tr}(h).
\end{aligned}
$$

So $\varrho_1\varrho_2\mathrm{Tr}(h), (\varrho_1+\varrho_2)\mathrm{Tr}(h) \in G$. Show that $\varrho\sigma(h) \in S$. Identifying some variables of h, we can obtain a function $(\sigma(h)+1)x+\beta(h)$, hence, $\sigma(h)+1 \in \mathrm{K}(\mathbf{N}_1 \cup \mathbf{N}_2)$. Thus, $\varrho\sigma(h) \subseteq \varrho\mathrm{K}(\mathbf{N}_1 \cup \mathbf{N}_2) - \varrho$. By Claim 1 the inclusion $\varrho\mathrm{K}(\mathbf{N}_1 \cup \mathbf{N}_2) \subseteq S$ implies $\varrho\sigma(h) \in S$. So, $\mathrm{Tr}(g) *_\varrho \mathrm{Tr}(h) \in G$.

Now, let us examine the inductive step. If $h = \theta(h')$ where $\theta \in \{\tau, \xi, \nabla, \Delta\}$ and $\mathrm{Tr}(g) *_\varrho \mathrm{Tr}(h') \in G$, then the equality $\mathrm{Tr}(h) = \mathrm{Tr}(h')$ implies $\mathrm{Tr}(g) *_\varrho \mathrm{Tr}(h) \in G$. Suppose that $h = h_1 \circ h_2$ where $h_1, h_2 \in \mathrm{F}_1 \vee \mathrm{F}_2$ and $\mathrm{Tr}(g \circ h_1), \mathrm{Tr}(g \circ h_2) \in G$ for all $g \in \mathbf{Q}'$. Then $g \circ h = g \circ (h_1 \circ h_2) = (g \circ h_1) \circ h_2$. Since $(g \circ h_1) \circ h_2$ is primitive the first variable of $g \circ h_1$ is non-invertible. So, $\mathrm{Tr}(g \circ h_1) \in G$ implies $(g \circ h_1) \circ h_2 \in \mathbf{Q}'$ and, consequently, $\mathrm{Tr}((g \circ h_1) \circ h_2) \in G$. By Lemma 2 we may assume that the first coefficient of g is ϱ. Hence $\mathrm{Tr}(g \circ h) = \mathrm{Tr}(g) *_\varrho \mathrm{Tr}(h) \in G$. This completes the proof of the claim and of Proposition 3. □

2.2. Intervals $\mathrm{LNP}(\mathbf{Q})$

If the ring R is not a direct sum of fields, i.e. R contains a nilpotent element, then among the intervals $\mathrm{LNP}(\mathbf{Q})$ there are both finite and infinite ones. In this section we characterize primitive algebras $\mathbf{Q}$ whose interval $\mathrm{LNP}(\mathbf{Q})$ is finite, and describe $\mathrm{LNP}(\mathbf{Q})$ in this case.

The idealizer of a subring S is denoted by $\mathrm{I}(S)$.

Lemma 9 *Let $\mathbf{Q}$ be a primitive algebra and $S = \mathrm{Nil}\,(\mathbf{Q})$, $G = \mathrm{Tr}(\mathbf{Q})$. If S satisfies the conditions*

(a) $\mathrm{I}(S) \cap \mathrm{Nil}\,(R) \supset S$;

(b) there exists a pair (α, b) where $\alpha \in \mathrm{I}(S) - 1$, $b \in M$, such that $(\varrho\alpha, \varrho b) \in G$ for any $\varrho \in S$,

then $\mathrm{LNP}(\mathbf{Q})$ is infinite.

Proof.

Let $\varrho \in (\mathrm{I}(S) \backslash S) \cap \mathrm{Nil}\,(R)$ be such that $\varrho^2 \in S$. For every $n \in \mathbb{N}$, we put

$$f_n = \sum_{i=1}^{n} \varrho x_i - \sum_{i=1}^{n} \varrho y_i + (\alpha+1)z + b.$$

For any $\varrho' \in S$ and $n \in \mathbb{N}$, we have $\varrho'\mathrm{Tr}(f_n) = \varrho'(\alpha, b) \in G$ by the condition (b). Hence f_n belongs to $\mathbf{N}(\mathbf{Q})$.

Let $\mathbf{N}_k = \langle\{f_k\} \cup \mathbf{Q}\rangle$ where $k \in \mathbb{N}$. Obviously, $\mathbf{N}_k \in \mathrm{LNP}(\mathbf{Q})$. Prove that if $k < j$, then $\mathbf{N}_k \neq \mathbf{N}_j$. We verify that $f_j \notin \mathbf{N}_k$. For any $g \in \mathbf{N}_k$, there are at most $2k+1$ coefficients of g that do not belong to S. Indeed, f_k has $2k$ or $2k+1$ coefficients of this kind and a function from $\mathbf{Q}$ has no one. Further, take $f_1, f_2 \in \mathbf{N}_k$ such that f_1, f_2 have respectively n_1 and n_2 coefficients which do not belong to S. For every $\gamma \in \langle\{\varrho\} \cup S\rangle$, we have $\gamma^2 \in S$. So, if γ is a non-invertible coefficient of a function from $\mathbf{N}_k$, then $\gamma^2 \in S$. Using this fact, it is easy to see that functions $\tau(f_1)$, $\xi(f_1)$, $\Delta(f_1)$, $\nabla(f_1)$ and $f_1 \circ f_2$ have at most $\max(n_1, n_2)$ coefficients which do not belong to S.

To complete the proof we note that at least $2j > 2k+1$ coefficients of f_j do not belong to S.

Now it is easy to see that algebras $\mathbf{N}_k$ form an infinite ascending chain. $\square$

Let S be a subring of $\mathrm{Nil}\,(R)$ and let G be a subalgebra of G_S. We will denote by $H_{S,G}$ the set of all pairs $(\alpha, b) \in (S-1) \times M$ such that $(\varrho\alpha, \varrho b) \in G$ for every $\varrho \in S$. We consider $H_{S,G}$ as an algebra endowed with operations $\{*_\varrho \mid \varrho \in S\}$. To show that the set $H_{S,G}$ is closed under these operations let us take $(\alpha_1, b_1), (\alpha_2, b_2) \in H_{S,G}, \varrho, \varrho' \in S$. Then

$$\begin{aligned}\varrho'((\alpha_1, b_1) *_\varrho (\alpha_2, b_2)) &= (\varrho'\alpha_1 + \varrho'\varrho\alpha_2, \varrho' b_1 + \varrho'\varrho b_2) = \\ &= (\varrho'\alpha_1, \varrho' b_1) *_\varrho (\varrho'\alpha_2, \varrho' b_2) \in G.\end{aligned}$$

Let $\mathrm{Sub}'\ H_{S,G}$ denote the lattice of all subalgebras H of $H_{S,G}$ such that $(\alpha, b) *_\varrho (\alpha', b') \in H$ and $(\alpha', b') \diamond (\alpha, b) \in H$ for every $(\alpha, b) \in H$, $(\alpha', b') \in G$, and $\varrho \in S$.

Lemma 10 *Let* $\mathrm{F} \in \mathrm{LNP}(\mathbf{Q})$ *and* $f \in \mathrm{F} \cap \mathbf{N}_M$. *If* $g \in \mathbf{N}_M$ *is such that* $\mathrm{K}(g) \subseteq S$ *and* $\mathrm{Tr}(g) = \mathrm{Tr}(f)$, *then* $g \in \mathrm{F}$.

Proof.

Since $f \in \mathrm{F}$, we have $(\sigma(f)+1)x + \beta(f) \in \mathrm{F}$. Suppose $g = \sum_{i=1}^n \alpha_i x_i + \beta(f)$. By Lemma 1 the function $x - \alpha_i y + \alpha_i z$ belongs to $\mathbf{Q}$ for every $i \in \underline{n}$. Hence, the function $\overline{g} = (\sigma(f)+1)x - \sum_{i=1}^n \alpha_i y_i + \sum_{i=1}^n \alpha_i z_i + \beta(f)$ belongs to F. Finally, $g(x_1, \ldots, x_n) = \overline{g}(z, z, \ldots, z, x_1, \ldots, x_n) \in \mathrm{F}$ and Lemma 10 is proved. $\square$

Proposition 4 *Let* $\mathbf{Q}$ *be a primitive algebra and* $S = \mathrm{Nil}\,(\mathbf{Q})$*,* $G = \mathrm{Tr}(\mathbf{Q})$*. If* $\mathrm{I}(S) = S$*, then* $\mathrm{LNP}(\mathbf{Q}) \cong \mathrm{Sub}'\, H_{S,G}$*.*

Proof.

Since $\mathrm{K}(\mathbf{N}(\mathbf{Q})) \cdot S \subseteq S$ and $\mathrm{I}(S) = S$, we have $\mathrm{K}(\mathbf{N}(\mathbf{Q})) \subseteq S$. By Lemma 8 it follows that $\mathbf{N}(\mathbf{Q})$ is the set of all functions f such that all coefficients of f belong to S and $\mathrm{Tr}(f) \in H_{S,G}$.

Let $\psi : \mathrm{LNP}(\mathbf{Q}) \to \mathrm{Sub}'\ H_{S,G}$ be a mapping which takes an algebra $\mathrm{F} \in \mathrm{LNP}(\mathbf{Q})$ to the set $\{\mathrm{Tr}(f) \mid f \in \mathrm{F} \cap \mathbf{N}_M\}$. We prove that ψ is an isomorphism. To show that $\psi(\mathrm{F})$ is a subalgebra of $H_{S,G}$ take (α_1, b_1), $(\alpha_2, b_2) \in \psi(\mathrm{F})$, $(\alpha', b') \in G$ and $\varrho \in S$. There are $f, g \in \mathrm{F} \cap \mathbf{N}_M$ and $h = (\sigma(h)+1)x + \beta(h) \in \mathbf{Q}$ such that $\mathrm{Tr}(f) = (\alpha_1, b_1)$, $\mathrm{Tr}(g) = (\alpha_2, b_2)$ and $\mathrm{Tr}(h) = (\alpha', b')$. By Lemma 10 we may assume that the first coefficient of f is equal to ϱ. Therefore equalities

$$\begin{aligned}
\mathrm{Tr}(f \circ g) &= (\alpha_1 + \varrho\alpha_2, b_1 + \varrho b_2) = (\alpha_1, b_1) *_\varrho (\alpha_2, b_2),\\
\mathrm{Tr}(f \circ h) &= (\alpha_1 + \varrho\alpha', b_1 + \varrho b') = (\alpha_1, b_1) *_\varrho (\alpha', b'),\\
\mathrm{Tr}(h \circ f) &= (\alpha' + (\sigma(h)+1)\alpha_1, b' + (\sigma(h)+1)b_1) = (\alpha', b') \diamond (\alpha_1, b_1)
\end{aligned}$$

provide that $\psi(\mathrm{F})$ belongs to $\mathrm{Sub}'\ H_{S,G}$.

By Lemma 10 ψ is an injective meet homomorphism. To prove that ψ is surjective, take a subalgebra H of $H_{S,G}$ and put $\mathrm{F}_H = \{f \mid f \in \mathbf{Q}$ or both $f \in \mathbf{N}(\mathbf{Q})$ and $\mathrm{Tr}(f) \in H\}$. Lemma 6 implies that the set F_H is closed under $\tau, \xi, \Delta, \nabla$. Taking $f, g \in \mathrm{F}_H$ where $f = \sum_{i=1}^n \alpha_i x_i + b$, we have to consider four cases.

1. $f, g \in \mathbf{Q}$ or $f \in \mathbf{Q}$ and α_1 is a nilpotent element. Then $f \circ g \in \mathbf{Q} \subseteq \mathrm{F}_H$.
2. $f, g \in \mathbf{N}(\mathbf{Q})$. Then

$$\mathrm{Tr}(f \circ g) = (\sigma(f) + \alpha_1\sigma(g), \beta(f) + \alpha_1\beta(g)) = \mathrm{Tr}(f) *_{\alpha_1} \mathrm{Tr}(g) \in H,$$

since $\alpha_1 \in S$.

3. $f \in \mathbf{Q}$, $g \in \mathbf{N}(\mathbf{Q})$ and α_1 is invertible. In this case $\alpha_1 - 1 - \sigma(f) = -\sum_{i=2}^n \alpha_i \in S$, hence $\mathrm{Tr}(f \circ g) = \mathrm{Tr}(f) *_{\alpha_1} \mathrm{Tr}(g) = (\mathrm{Tr}(f) \diamond \mathrm{Tr}(g)) *_{\alpha_1 - 1 - \sigma(f)} \mathrm{Tr}(g) \in H$.
4. $f \in \mathbf{N}(\mathbf{Q})$ and $g \in \mathbf{Q}$. Then $\mathrm{Tr}(f \circ g) = \mathrm{Tr}(f) *_{\alpha_1} \mathrm{Tr}(g) \in H$.

Thus, F_H is a subalgebra and ψ, ψ^{-1} are surjective monotone mappings. Now the proof of the proposition is complete. □

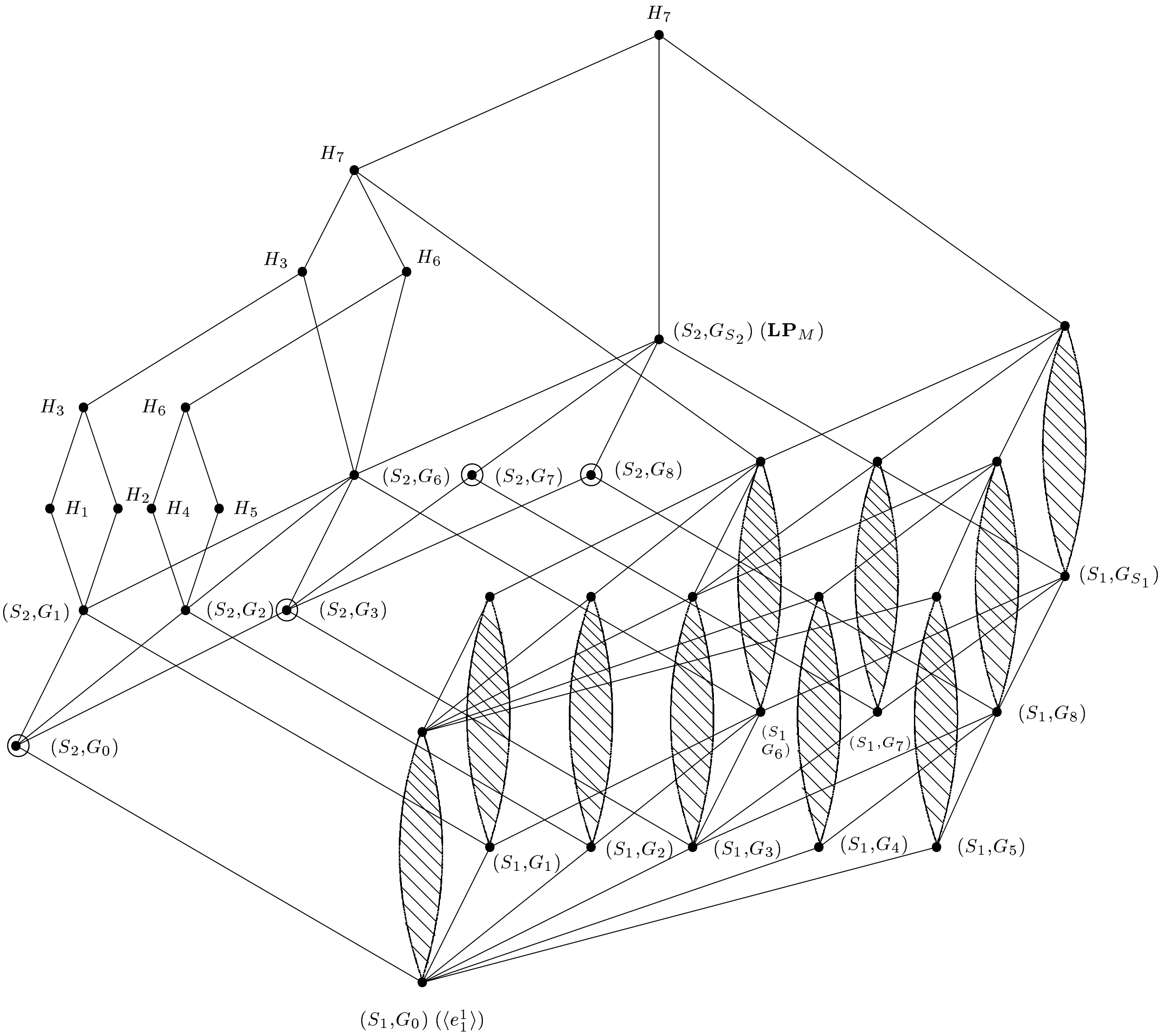

Fig. 2

$H_1 = \{(1,0),(3,0)\}$; $H_2 = \{(1,2),(3,2)\}$; $H_3 = H_{S_2,G_1} = \{(1,0),(1,2),(3,0),(3,2)\}$;
$H_4 = \{(1,1),(3,3)\}$; $H_5 = \{(1,3),(3,1)\}$; $H_6 = H_{S_2,G_2} = \{(1,1),(1,3),(3,1),(3,3)\}$;
$H_7 = H_{S_2,G_6} = \{(1,0),(1,1),(1,2),(1,3),(3,0),(3,1),(3,2),(3,3)\}$.

The lattice Sub $\mathbf{NP}_{\mathbb{Z}_4}$ is shown in Figure 2. Notation of primitive algebras is identical to Figure 1. A shaded

interval represents a lattice $\mathbb{LNP}(\mathbf{Q})$ when it is infinite. Encircled points correspond to primitive algebras $\mathbf{Q}$ for which $\mathbb{LNP}(\mathbf{Q})$ is singleton.

Obviously, $\mathbb{LNP}(\mathbf{Q})$ is singleton if and only if $\mathbf{N}(\mathbf{Q}) = \varnothing$, i.e. $H_{S,G} = \varnothing$. This implies the next statement.

Corollary 2 *Let $\mathbf{Q} \in \mathrm{Sub}\,\mathbf{LP}_M$. The lattice $\mathbb{LNP}(\mathbf{Q})$ is singleton if and only if condition (b) of Lemma 9 is not valid.*

When the interval $\mathbb{LNP}(\mathbf{Q})$ is infinite, its structure is closely connected with the one of the lattice of nilpotent algebras. However, this falls outside the limits of the paper.

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